MANUSCRIPT BOOK 1
OF
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BxM88
Mss. 3
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MADRAS.

$\sqrt{2} = 1.41421356237309504880 17$

~~$\sqrt{3} = 1.7320$~~

$3 — 64x^{24}$ $\qquad$ $7 — \frac{x^{24}}{64}$

$3 — x^3 = 4$

$11 — x^3 - x^2 + x = 4$

$19 — x^3 + x^2 = 4$

$27 — x^3 + x^2 \sqrt[3]{3} = 4$

35 —

$43 — x^3 + x = 4$

51 —

59 —

$67 — x^3 + x^2 + x = 4$

75 —

$7 — x^3 = 1$

15 —

$23 — x^3 + x^2 = 1$

$31 — x^3 + x = 1$

39 —

47

55

63

CHAPTER I.

MAGIC SQUARES.

Let <u>a</u> be the average, <u>s</u> a row or a column, <u>m</u> middle row or rows or column or columns, <u>d</u> a diagonal, and <u>W</u> the whole sum.

I. When the Square contains 3 rows and 3 columns,

i. If <u>s</u> and <u>d</u> are equal, write <u>a</u> in the middle and supply the other figures.

Sol:— $d_1 + d_2 + m_1 + m_2 = W + 3x$ where x is the reqd. figure in the middle.

$\therefore 4s = 3s + 3x \therefore s = 3x$ or $x = a$.

Cor. The figures in <u>d</u> are in A.P.

Sol:— The sum of the numbers in <u>d</u> is s or $3a$ and 2nd $= a$. $\therefore$ 1st + 3rd $= 2a$ = twice the second

$\therefore$ are in A.P. Similarly in <u>m</u> also.

Ex. 1. Fill up the Square when $S = 15$

6	1	8
7	5	3
2	9	4

2. When $s = 27$ and all numbers are odd.

15	1	11
5	9	13
7	17	3

ii. When s and d are unequal, write $\frac{d_1 + d_2 - S}{3}$ in the middle

Ex. 1. Show that the numbers in m are in A.P. here also

Sol. Proceed as in I. i. i cor.

Mss. 1

2. Fill up the Square when $S=20$, $d_1=16$ and $d_2=19$.

10	2	8
4	5	11
6	13	1

III. When the diagonals, columns and rows are all different, write $\frac{1}{3}(d_1+d_2+m_1+m_2-W)$ in the middle.

Sol. As in I 1.1.

Ex. Fill up the Square when $d_1=15$, $d_2=19$ and the columns and rows are 16, 17, 12, 6, 21, and 18.

1	2	3
8	9	4
7	6	5

2. When an oblong contains 3 rows and 4 columns,

A+C=2B+3D			
A	C+D	A+2D	C+3D
B+6D	B+4D	B+2D	B
C	A+D	C+2D	A+3D

Ex. Fill up the oblong when $a=8$

1	12	3	15
11	[illegible]	[illegible]	5
[illegible]	2	14	4

3. When a square contains 4 rows and 4 columns.

i. When the diagonals, columns and rows are all different, arrange the central four so that the sum may be $\frac{1}{2}(d_1+d_2+m_1+m_2-W)$.

ii. When s = d,

Fig I

A+P	C+S	D+Q	B+R
D+R	B+Q	A+S	C+P
B+S	D+P	C+R	A+Q
C+Q	A+R	B+P	D+S

Fig II

A+D = B+C		P+S = Q+R	
A+P	D+R	D+R	A+S
B+S	C+R	C+R	B+P
C+S	B+R	B+R	C+P
D+P	A+R	A+R	B+S

Ex.

1	10	15	8
16	7	2	9
6	13	12	3
11	4	5	14

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

4. When a square contains 5 rows and 5 columns.

A+P	E+R	D+T	C+Q	B+S
C+T	B+Q	A+S	E+P	D+R
E+S	D+P	C+R	B+T	A+Q
B+R	A+T	E+Q	D+S	C+P
D+Q	C+S	B+P	A+R	E+T

1	58	59	4	5	62	63	8
16	55	54	13	12	51	50	9
24	47	46	21	20	43	42	17
25	34	35	28	29	38	39	32
33	26	27	36	37	30	31	40
48	23	22	45	44	19	18	41
56	15	14	53	52	11	10	49
57	2	3	60	61	6	7	64

Similarly we can form squares containing more rows & columns.

CHAPTER II

1. $\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\dots\dots+\frac{1}{2n}$

$=\frac{1}{2^3-2}+\frac{1}{4^3-4}+\frac{1}{6^3-6}+\dots\dots+\frac{1}{(2n)^3-2n}$.

Sol:— $\frac{1}{(2n)^3-2n}=\frac{1}{2}\cdot\frac{1}{2n-1}-\frac{1}{2n}+\frac{1}{2(2n+1)}$.

$\therefore$ R.H.S $=\frac{1}{2}\left(1+\frac{1}{3}+\frac{1}{5}+\dots+\frac{1}{2n-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\dots\frac{1}{2n}\right)$

$+\frac{1}{2}\left(1+\frac{1}{3}+\frac{1}{5}+\dots+\frac{1}{2n+1}\right)+\frac{n}{2n+1}-\frac{1}{2}$.

$=\left(1+\frac{1}{3}+\frac{1}{5}+\dots+\frac{1}{2n-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\dots+\frac{1}{2n}\right)$

$=\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{2n}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{n}\right)$

$=\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\dots+\frac{1}{2n}$.

Cor. $2\log_e 2=1+\frac{2}{2^3-2}+\frac{2}{4^3-4}+\frac{2}{6^3-6}+\frac{2}{8^3-8}+\dots$ ad inf.

Sol. R.H.S $=2\left(\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}\right)$ when $n=\infty$

Let $n=\frac{1}{dx}$

then the given Series $=\frac{2dx}{1+dx}+\frac{2dx}{1+2dx}+\dots+\frac{2dx}{1+1}$

$=2\int_1^2\frac{1}{x}\,dx=2\log_e 2$.

or thus —

In the Solution of II 1. we got $\left(1+\frac{1}{3}+\frac{1}{5}+\dots\frac{1}{2n-1}\right)$

$-\left(\frac{1}{2}+\frac{1}{4}+\dots\frac{1}{2n}\right)$. When $n=\infty$ this becomes

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-$ &c $=\log_e 2$ $\therefore$ The req.d Sum $=2\log_e 2$.

N.B. $\Sigma\frac{1}{n}$ means the sum of the reciprocals of n natural numbers. Therefore $\Sigma\frac{1}{an}=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{an}$ and $\neq$ $\frac{1}{a}+\frac{1}{2a}+\frac{1}{3a}+\dots+\frac{1}{na}$. $\Sigma\frac{1}{m/n}$ should not be written as $\Sigma\frac{n}{m}$ which has no meaning according to our convention.

Ex. Shew that $\frac{n-1}{n+1}+\frac{n-2}{n+2}+\frac{n-3}{n+3}+\dots+\frac{n-n}{n+n}$

$=2n\left\{\frac{1}{1\cdot2\cdot3}+\frac{1}{3\cdot4\cdot5}+\frac{1}{5\cdot6\cdot7}+\dots+\frac{1}{(2n-1)(2n)(2n+1)}\right\}-\frac{n}{2n+1}$

Sol. We have by II 1.

$$\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}=\frac{n}{2n+1}+\frac{1}{1.2.3}+\frac{1}{3.4.5}+\dots\dots$$
$$+\frac{1}{(2n-1)(2n)(2n+1)}.$$ Multiplying both sides by $2n$

$$\frac{2n}{n+1}+\frac{2n}{n+2}+\dots+\frac{2n}{2n}=\frac{2n^2}{2n+1}+2n\left\{\frac{1}{1.2.3}+\frac{1}{3.4.5}+\&c \text{ to } n \text{ terms}\right\}.$$

$$\therefore \left(\frac{2n}{n+1}-1\right)+\left(\frac{2n}{n+2}-1\right)+\left(\frac{2n}{n+3}-1\right)+\dots+\left(\frac{2n}{2n}-1\right)$$
$$=\frac{2n^2}{2n+1}-n+2n\left\{\frac{1}{1.2.3}+\frac{1}{3.4.5}+\&c \text{ to } n \text{ terms}\right\}$$

$$\therefore \frac{n-1}{n+1}+\frac{n-2}{n+2}+\frac{n-3}{n+3}+\dots+\frac{n-n}{n+n}$$
$$=2n\left\{\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\dots+\frac{1}{(2n-1)2n(2n+1)}\right\}-\frac{n}{2n+1}$$

2. $$\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{3n+1}=1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\dots+\frac{2}{(3n)^3-3n}$$

Sol. By proceeding as in II 1 we have R.H.S $=\Sigma\frac{1}{3n+1}-\Sigma\frac{1}{n}=$ L.H.S

Cor. $1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\frac{2}{9^3-9}+\&c=\log_e 3.$

3. $$\tan^{-1}\frac{1}{n+1}+\tan^{-1}\frac{1}{n+2}+\dots+\tan^{-1}\frac{1}{3n+1}$$
$$=\tan^{-1}1+\tan^{-1}\frac{10}{5.8}+\tan^{-1}\frac{20}{14.35}+\dots+\tan^{-1}\frac{10n}{(3n^2+2)(9n^2}$$

Cor. $\log_e 3=\tan^{-1}1+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{49}+\tan^{-1}\frac{3}{232}+\tan^{-1}\frac{4}{715}+\&$

4. $$\left(\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}\right)+\left(\frac{1}{2n+1}+\frac{1}{2n+3}+\dots\frac{1}{4n+1}\right)$$
$$=1+\frac{2}{4^3-4}+\frac{2}{8^3-8}+\frac{2}{12^3-12}+\dots+\frac{2}{(4n)^3-4n}$$
$$=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{4n+1}\right)+\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots-\frac{1}{2n}\right)$$

Sol. By proceeding as in II 1 we have R.H.S $=\Sigma\frac{1}{4n+1}-\frac{1}{2}\Sigma\frac{1}{2n}$
$$-\frac{1}{2}\Sigma\frac{1}{n}=\Sigma\frac{1}{4n+1}-\Sigma\frac{1}{n}-\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}\right)$$
$$=\left(\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{4n+1}\right)-\left(\frac{1}{2n+2}+\frac{1}{2n+4}+\dots+\frac{1}{4n}\right)$$
$$=\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\dots+\frac{1}{2n}\right)+\left(\frac{1}{2n+1}+\frac{1}{2n+3}+\dots+\frac{1}{4n+1}\right)$$

Again $\sum \frac{1}{4n+1} - \frac{1}{2}\sum \frac{1}{2n} - \frac{1}{2}\sum \frac{1}{n} = \sum \frac{1}{4n+1} - \sum \frac{1}{2n} + \frac{1}{2}\sum \frac{1}{2n}$

$-\frac{1}{2}\sum \frac{1}{n} = \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{4n+1}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{4n}\right)$

$+\frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2n}\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n}\right)$

$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{4n+1}$

$+ \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{2n}\right).$

Cor. $1 + \frac{2}{4^3-4} + \frac{2}{8^3-8} + \frac{2}{12^3-12} + \&c = \frac{3}{2}\log_e 2.$

5. $\frac{2}{3}\left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}\right) + \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \dots + \frac{1}{6n+1}\right)$

$= 1 + \frac{2}{6^3-6} + \frac{2}{12^3-12} + \frac{2}{18^3-18} + \dots + \frac{2}{(6n)^3-6n}$

Sol. Proceeding as in III the Sum is $\sum \frac{1}{6n+1} - \frac{1}{2}\sum \frac{1}{3n}$

$-\frac{1}{3}\sum \frac{1}{2n} - \frac{1}{6}\sum \frac{1}{n} =$ L.H.S.

Cor. $1 + \frac{2}{6^3-6} + \frac{2}{12^3-12} + \frac{2}{18^3-18} + \&c = \frac{1}{2}\log_e 3 + \frac{1}{3}\log_e 4$

Ex. 1. $\frac{1}{4}\log_e 2 = \frac{1}{2^3-2} + \frac{1}{6^3-6} + \frac{1}{10^3-10} + \frac{1}{14^3-14} + \&c$

2. $\log_e 2 = 1 - \frac{2}{2^3-2} + \frac{2}{4^3-4} - \frac{2}{6^3-6} + \&c$

3. $2\left\{1 + \frac{2}{4^3-4} + \frac{2}{8^3-8} + \frac{2}{12^3-12} + \dots + \frac{2}{(4n)^3-4n}\right\}$

$= \left\{1 + \frac{2}{2^3-2} + \frac{2}{4^3-4} + \dots + \frac{2}{(4n)^3-4n}\right\} + \frac{1}{(4n+1)(4n+2)}$

$+ \frac{1}{2}\left\{1 + \frac{2}{2^3-2} + \frac{2}{4^3-4} + \frac{2}{6^3-6} + \dots + \frac{2}{(2n)^3-2n}\right\}$

4. Shew that $1 + \frac{2}{4^3-4} + \frac{2}{8^3-8} + \dots + \frac{2}{(4n)^3-4n}$

$= \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2} + \dots \frac{1}{4n}\right) + \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \dots \frac{1}{4n+1}\right)$

5. $\frac{1}{3^3-3} + \frac{1}{9^3-9} + \frac{1}{15^3-15} + \&c = \frac{1}{4}\log_e 3 - \frac{1}{3}\log_e 2.$

6. $\frac{4}{3}\log_e 2 = 1 - \frac{2}{3^3-3} + \frac{2}{6^3-6} - \frac{2}{9^3-9} + \&c$

7. Shew that $2\left\{1+\frac{2}{6^3-6}+\frac{2}{12^3-12}+\cdots+\frac{2}{(6n)^3-6n}\right\}$

$+\frac{1}{3}\left\{1+\frac{2}{2^3-2}+\frac{2}{4^3-4}+\frac{2}{6^3-6}+\cdots+\frac{2}{(2n)^3-2n}\right\}$

$=1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\cdots+\frac{2}{(3n)^3-3n}+\frac{2}{(6n+1)(6n+2)(6n+3)}$

$+1+\frac{2}{2^3-2}+\frac{2}{4^3-4}+\cdots+\frac{2}{(6n)^3-6n}$.

8. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{6}+\tan^{-1}\frac{1}{7}$

$+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{9}+\tan^{-1}\frac{1}{10}+\tan^{-1}\frac{1}{11}+\tan^{-1}\frac{1}{12}+\tan^{-1}\frac{1}{13}$

$=\frac{\pi}{2}+2\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{49}+\tan^{-1}\frac{3}{232}+\tan^{-1}\frac{4}{715}$

9. $2\left(\tan^{-1}\frac{1}{n+1}+\tan^{-1}\frac{1}{n+2}+\cdots\tan^{-1}\frac{1}{2n+1}\right)=\tan^{-1}\frac{n+1}{n}$

$+\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{4}{137}+\tan^{-1}\frac{6}{667}+\tan^{-1}\frac{8}{2081}+$

$\cdots+\tan^{-1}\frac{2n}{8n^4+2n^2+1}+$

$2\left(\tan^{-1}\frac{1}{1.7}+\tan^{-1}\frac{1}{2.19}+\tan^{-1}\frac{1}{3.39}+\cdots+\tan^{-1}\frac{1}{n(4n^2+3)}\right)$

10. $\tan^{-1}\frac{1}{n+1}+\tan^{-1}\frac{1}{n+2}+\cdots+\tan^{-1}\frac{1}{2n+1}+\tan^{-1}\frac{1}{2n+3}$

$+\tan^{-1}\frac{1}{2n+5}+\cdots+\tan^{-1}\frac{1}{4n+1}$

$=\frac{\pi}{4}+\tan^{-1}\frac{9}{53}+\tan^{-1}\frac{18}{599}+\tan^{-1}\frac{27}{2789}+\cdots+\tan^{-1}\frac{9n}{32n^4+22n^2-1}$

$+\tan^{-1}\frac{4}{137}+\tan^{-1}\frac{8}{2081}+\tan^{-1}\frac{12}{10441}+\cdots\tan^{-1}\frac{4n}{128n^4+8n^2+1}$

N.B. $1+\frac{2}{x^3-x}+\frac{2}{(2x)^3-2x}+\frac{2}{(3x)^3-3x}+\cdots+\frac{2}{(nx)^3-nx}$ cannot be expressed as in II 2. for all values of x but 2, 3, 4 and 6 though it can be summed up for all values of x when $n=\infty$. Refer to Chapter

6. If $K_n=3^n(n+\frac{1}{2})-\frac{1}{2}$, then

$\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{K_n}=$

i $B_n - \frac{n(n-1)}{2.3} B_{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.3.4.5} B_{n-4} - \&c$ iii

ii $B_n - \frac{n(n-1)}{3.4} B_{n-2} + \frac{n(n-1)(n-2)(n-3)}{3.4.5.6} B_{n-4} - \&c + \frac{(-1)^{\frac{n}{2}}}{2} = 0$

iii $B_n - \frac{n(n-1)}{4.6} B_{n-2} + \frac{n(n-1)(n-2)(n-3)}{4.6.8.10} B_{n-4} - \&c + \frac{(-1)^{\frac{n}{2}}}{n+1} = 0$

N.B. 0 is excluded in i.

$+ \frac{(-1)^{\frac{n}{2}}}{2^n} = 0$

~~$B_n = B_{n-2} \left\{ \frac{n(n-1)}{40} + 33\, n(n-1) \times .000100090 \right\}$~~

~~$\log_e B_n = \log_e B_{n-2} + \log n + \log(n-1) - 2 \log 2\pi$~~

$B_n = \frac{n(n-1)}{4\pi^2} B_{n-2} \left(1 - \frac{3}{2^n-1}\right)\left(1 - \frac{8}{3^n-1}\right)\left(1 - \frac{24}{5^n-1}\right)\left(1 - \frac{48}{7^n-1}\right)$

$B_n = \frac{n(n-1)}{(n-2)(n-3)} \frac{B_{n-2} \cdot B_{n-2}}{B_{n-4}} \Pi \left\{1 + \frac{(p^2-1)^2}{(p^n - 1 - p^4 + \frac{p^4}{p^n})}\right\}$

$(n + \tfrac{1}{2}) \log_{10} n - 1.2324743503\, n + .700120$

$+ \frac{.0362}{n}$

$4 \left\{ B_2 \frac{x^2}{\lfloor 2} + B_6 \frac{x^6}{\lfloor 6} + B_{10} \frac{x^{10}}{\lfloor 10} + \&c \right\}$

$= x. \frac{\frac{x^3}{\lfloor 3} - \frac{x^7}{2^2 \lfloor 7} + \frac{x^{11}}{2^4 \lfloor 11} - \&c}{\frac{x^2}{\lfloor 2} - \frac{x^6}{2^2 \lfloor 6} + \frac{x^{10}}{2^4 \lfloor 10} - \&c}$

Sin

$\frac{x}{\lfloor 7}$

~~Sin~~

$2 \left\{ B_0 + B_4 \frac{x^4}{\lfloor 4} + B_8 \frac{x^8}{\lfloor 8} + \&c \right\}$

$= -x. \frac{\frac{x}{\lfloor 1} - \frac{x^5}{2^2 \lfloor 5} + \frac{x^9}{2^4 \lfloor 9} - }{\frac{x^2}{\lfloor 2} - \frac{x^6}{2^2 \lfloor 6} + \frac{x^{10}}{2^4 \lfloor 10} - \&c}$

Sanitary Engineer office Chap. I
Accountant General's office
Postal audit office
(Cat. Comptroller of Post offices)

$$e^{-2\pi} = \frac{1}{540}\left[1 + \left\{\frac{1}{120}\right.\right.$$

$$= \frac{1}{540}\left[1 + \frac{1}{120}\left\{1 + \frac{1}{100}\left(1 + \frac{1}{35}\cdot\overline{1 + \frac{1}{90} + \frac{1}{90\cdot 25}}\right)\right\}\right]$$

$$x\cot x = 1 - B_2\frac{(2x)^2}{\lfloor 2} - B_4\frac{(2x)^4}{\lfloor 4} - B_6\frac{(2x)^6}{\lfloor 6} - \&c$$

(a) $$-x\left\{\cot\frac{x}{2} + \cot\frac{x\omega}{2} + \cot\frac{x\omega^2}{2}\right\}$$

$$= 6\left\{\frac{B_4}{\lfloor 4}x^4 + \frac{B_{10}}{\lfloor 10}x^{10} + \frac{B_{16}}{\lfloor 16}x^{16} + \&c\right\}$$

$$= x.\frac{\frac{x^6}{\lfloor 6} - \frac{x^{12}}{\lfloor 12} + }{\frac{x^3}{\lfloor 3} - \frac{x^9}{\lfloor 9} + \&c}$$

(b) $$-\frac{x}{2}\left\{\cot\frac{x}{2} + \omega^2\cot\frac{x\omega}{2} + \omega\cot\frac{x\omega^2}{2}\right\}$$

$$= 3\left\{\frac{B_2}{\lfloor 2}x^2 + \frac{B_8}{\lfloor 8}x^8 + \&c\right\}$$

$$= x.\frac{\frac{x^4}{\lfloor 4} - \frac{x^{10}}{\lfloor 10} + \&c}{\frac{x^3}{\lfloor 3} - \frac{x^9}{\lfloor 9} + \&c}$$

(c) $$-\frac{x}{2}\left\{\cot\frac{x}{2} + \omega\cot\frac{x\omega}{2} + \omega^2\cot\frac{x\omega^2}{2}\right\}$$

$$= 3\left\{B_0 + \frac{B_6}{\lfloor 6}x^6 + \frac{B_{12}}{\lfloor 12}x^{12} + \&c\right\}$$

$$= -x.\frac{\frac{x^2}{\lfloor 2} - \frac{x^8}{\lfloor 8} + \&c}{\frac{x^3}{\lfloor 3} - \frac{x^9}{\lfloor 9} + \&c}$$

$$B_n - \frac{n}{4}.\frac{n-1}{5}\ldots\frac{n-5}{9}B_{n-6} + \frac{n}{4}.\frac{n-1}{5}\ldots\frac{n-11}{15}B_{n-12}$$

$$+ \&c \quad ~~\text{as f}~~ = \frac{2}{(n+1)(n+2)}(-1)^{\frac{n-2}{6}} \text{ or}$$

$$\frac{2}{(n+1)(n+2)}(-1)^{\frac{n-6}{6}} \text{ or}$$

$$\frac{1}{(n+1)(n+2)}(-1)^{\frac{n-4}{6}}$$

$B_2 = \frac{1}{6}$

$B_8 - \frac{B_2}{3} = -\frac{1}{45}$

$B_{14} - \frac{143}{4} B_8 + \frac{B_2}{5} = \frac{1}{120}$

$B_4 = \frac{1}{30}$

$B_{10} - \frac{5}{2} B_4 = -\frac{1}{132}$

$B_{16} - \frac{286}{3} B_{10} + 4 B_4 = \frac{1}{306}$

$B_0 = -1$

$B_6 - \frac{B_0}{84} = \frac{1}{28}$

$B_{12} - 11 B_6 + \frac{B_0}{455} = -\frac{1}{91}$

$B_{18} - 221 B_{12} + \frac{204}{5} B_6 - \frac{B_0}{1330} = \frac{1}{190}$

$\frac{n+2}{3} B_n = {}^nC_6 B_{n-6} B_6 + {}^nC_{12} B_{n-12} B_n$ +&

when $n-2$ is a multiple of 6.

$$n\left\{1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\frac{2}{9^3-9}+\cdots+\frac{2}{(3n)^3-3n}\right\}$$

$$(n-1)\left\{\frac{2}{(3K_0+3)^3-(3K_0+3)}+\frac{2}{(3K_0+6)^3-(3K_0+6)}+\cdots+\frac{2}{(3K_1)^3-3K_1}\right\}$$

$$+(n-2)\left\{\frac{2}{(3K_1+3)^3-(3K_1+3)}+\frac{2}{(3K_1+6)^3-(3K_1+6)}+\cdots+\frac{2}{(3K_2)^3-3K_2}\right\}$$

$+\cdots$ to n terms.

Cl. By II 2 we have,

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}=1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\cdots+\frac{2}{(3n)^3-3n}$$

$$\frac{1}{3n+2}+\frac{1}{3n+3}+\cdots+\frac{1}{9n+4}=1+\frac{2}{3^3-3}+\cdots+\frac{2}{(9n+3)^3-(9n+3)}$$

$$\frac{1}{9n+5}+\frac{1}{9n+6}+\cdots+\frac{1}{27n+13}=1+\frac{2}{3^3-3}+\cdots+\frac{2}{(27n+12)^3-(27n+12)}$$

writing thus n terms and then adding up all the terms we can get the result.

Ex. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\cdots+\frac{1}{\frac{3^n-1}{2}}$

$$=n+(n-1)\left(\frac{2}{3^3-3}\right)+(n-2)\left(\frac{2}{6^3-6}+\frac{2}{9^3-9}+\frac{2}{12^3-12}\right)$$

$$+(n-3)\left(\frac{2}{15^3-15}+\frac{2}{18^3-18}+\cdots+\frac{2}{39^3-39}\right)+ \text{\&c to } n \text{ terms.}$$

V. 13. II 6 & II 6 Cor. are very useful in finding the approximate value of $\Sigma\frac{1}{n}$ whether n is small or very great without knowing logarithms, differential and integral calculus. In finding $\Sigma\frac{1}{n}$ it must be remembered that when a_1 and a_n are very great and a_1, a_2, a_3 &c are in A.P. the approximate value of $\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}$

$$=\frac{2n}{a_1+a_n}.$$

Ex. 1. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}$

$$=3+\frac{1}{6}+\frac{1}{105}+\frac{1}{360}+\frac{1}{858}.$$

2. Show that $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{1000}$
$= 7\frac{1}{2}$ very nearly.

7. $\tan^{-1}\frac{2}{(n+1)^2} + \tan^{-1}\frac{2}{(n+3)^2} + \tan^{-1}\frac{2}{(n+5)^2} + \&c$ to r terms

$= \tan^{-1}\frac{2r}{n^2+2nr+1}$.

Sol. $\tan^{-1}\frac{1}{n} - \tan^{-1}\frac{1}{n+2} = \tan^{-1}\frac{2}{(n+1)^2}$

$\therefore$ L.H.S $= \left(\tan^{-1}\frac{1}{n} - \tan^{-1}\frac{1}{n+2}\right) + \left(\tan^{-1}\frac{1}{n+2} - \tan^{-1}\frac{1}{n+4}\right)$

$+ \left(\tan^{-1}\frac{1}{n+4} - \tan^{-1}\frac{1}{n+6}\right) + \dots + \left(\tan^{-1}\frac{1}{n+2r-2} - \tan^{-1}\frac{1}{n+2r}\right)$

$= \tan^{-1}\frac{1}{n} - \tan^{-1}\frac{1}{n+2r} = \tan^{-1}\frac{2r}{n^2+2nr+1}$.

Cor. $\tan^{-1}\frac{2}{(n+1)^2} + \tan^{-1}\frac{2}{(n+3)^2} + \tan^{-1}\frac{2}{(n+5)^2} + \&c = \tan^{-1}\frac{1}{n}$.

Sol. Make r infinite in II 7.

Ex. 1. $\tan^{-1}\frac{2}{(n+1)^2} + \tan^{-1}\frac{2}{(n+2)^2} + \tan^{-1}\frac{2}{(n+3)^2} + \&c = \tan^{-1}\frac{2n+1}{n^2+n}$

Sol. $\tan^{-1}\frac{2}{(n+1)^2} + \tan^{-1}\frac{2}{(n+3)^2} + \&c = \tan^{-1}\frac{1}{n}$

$\therefore \tan^{-1}\frac{2}{(n+2)^2} + \tan^{-1}\frac{2}{(n+4)^2} + \&c = \tan^{-1}\frac{1}{n+1}$

$\therefore \tan^{-1}\frac{2}{(n+1)^2} + \tan^{-1}\frac{2}{(n+2)^2} + \&c = \tan^{-1}\frac{2n+1}{n^2+n-1}$.

N.B. If $n < \frac{\sqrt{5}-1}{2}$ add π to R.H.S.

2. $\tan^{-1}\frac{2}{(n+1)^2} - \tan^{-1}\frac{2}{(n+2)^2} + \tan^{-1}\frac{2}{(n+3)^2} - \&c = \tan^{-1}\frac{1}{n^2+n+1}$

3. $\tan^{-1}\frac{1}{2(n+1)^2} + \tan^{-1}\frac{1}{2(n+2)^2} + \tan^{-1}\frac{1}{2(n+3)^2} + \&c = \tan^{-1}\frac{1}{2n+1}$

4. $\frac{3\pi}{4} = \tan^{-1}\frac{2}{1^2} + \tan^{-1}\frac{2}{2^2} + \tan^{-1}\frac{2}{3^2} + \&c$

5. $\frac{\pi}{4} = \tan^{-1}\frac{1}{2\cdot1^2} + \tan^{-1}\frac{1}{2\cdot2^2} + \&c = \tan^{-1}\frac{2}{1^2} - \tan^{-1}\frac{2}{2^2} + \&c$

6. $\frac{\pi}{8} = \tan^{-1}\frac{1}{(1+\sqrt{2})^2} + \tan^{-1}\frac{1}{(1+2\sqrt{2})^2} + \tan^{-1}\frac{1}{(1+3\sqrt{2})^2} + \&c$

8. If $1 = A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + \&c$

and $\begin{cases} P_n = A_1 P_{n-1} + A_2 P_{n-2} + A_3 P_{n-3} + \&c \cdots + A_{n-1} P_1. \\ Q_n = A_1 Q_{n-1} + A_2 Q_{n-2} + A_3 Q_{n-3} + \cdots\cdots + A_{n-1} Q_1 + A_n Q_0. \\ P_1 = 1 \text{ and } Q_0 = 1 \end{cases}$

then $\frac{P_n}{Q_n}$ approaches x when n becomes greater & greater.

Eg. 1. $x + x^2 = 1$

$x = \frac{0}{1}, \frac{1}{1}, \Big| \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \&c\&c$

2. $x + x^2 + x^3 = 1$

$x = \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \Big| \frac{2}{4}, \frac{4}{7}, \frac{7}{13}, \frac{13}{24}, \frac{24}{44}, \&c$

3. $x + x^3 = 1$

$x = \frac{0}{1}, \frac{1}{1}, \frac{1}{1}, \Big| \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{6}, \frac{6}{9}, \frac{9}{13}, \frac{13}{19}, \&c$

N.B. If $\frac{p}{q}$ and $\frac{r}{s}$ are two consecutive convergents to x then we may take $\frac{mp + nr}{mq + ns}$ in a suitable manner equivalent to x.

Ex. 1. Find convergents to $\log_e 2$.

Sol. Let $\log_e 2 = x$ then $e^x = 2$

$\therefore 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \&c$

$\therefore x = \frac{0}{1}, \frac{1}{1}, \Big| \frac{1}{1\frac{1}{2}}, \frac{1\frac{1}{2}}{2\frac{1}{6}}, \frac{2\frac{1}{6}}{3\frac{1}{8}}, \frac{3\frac{1}{8}}{4\frac{61}{120}} \&c$

$= \frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{9}{13}, \frac{52}{75}, \frac{375}{541} \&c$

2. When $e^{-x} = x$ shew that the convergents to x are

$\frac{1}{2}, \frac{4}{7}, \frac{21}{37}, \frac{148}{261}, \&c.$

If $\phi(x) = e^x \psi(x)$, then

$$\phi(0)f(0) + \frac{\phi'(0)f'(0)}{\lfloor\underline{1}} + \frac{\phi''(0)f''(0)}{\lfloor\underline{2}} + \&c$$

$$= \psi(0)f(1) + \frac{\psi'(0)f'(1)}{\lfloor\underline{1}} + \frac{\psi''(0)f''(1)}{\lfloor\underline{2}} + \&c$$

$$\frac{f(0)}{n} + \frac{f'(0)}{(n+1)\lfloor\underline{1}} + \frac{f''(0)}{(n+2)\lfloor\underline{2}} + \&c$$

$$= \frac{f(1)}{n} - \frac{f'(1)}{n(n+1)} + \frac{f''(1)}{n(n+1)(n+2)} - \&c$$

$$e^x \left\{ \frac{x}{1\lfloor\underline{1}} - \frac{2x^2}{2\lfloor\underline{2}} + \frac{2^2x^3}{3\lfloor\underline{3}} - \frac{2^3x^4}{4\lfloor\underline{4}} + \&c \right\}$$

$$= \frac{x}{\lfloor\underline{1}} + \frac{x^2}{\lfloor\underline{2}} + \frac{x^3}{\lfloor\underline{3}}\left(1+\tfrac{1}{3}\right) + \frac{x^4}{\lfloor\underline{4}}\left(1+\tfrac{1}{3}\right) + \frac{x^5}{\lfloor\underline{5}}\left(1+\tfrac{1}{3}+\tfrac{1}{5}\right)$$

$$+ \frac{x^6}{\lfloor\underline{6}}\left(1+\tfrac{1}{3}+\tfrac{1}{5}\right) + \&c$$

CHAPTER III

Let $\frac{1^n}{\underline{|0}}x + \frac{2^n}{\underline{|1}}x^2 + \frac{3^n}{\underline{|2}}x^3 + \frac{4^n}{\underline{|3}}x^4 + \&c = e^x f_n(x)$.

1. $$\frac{x}{n\underline{|0}} + \frac{x^2}{(n+1)\underline{|1}} + \frac{x^3}{(n+2)\underline{|2}} + \frac{x^4}{(n+3)\underline{|3}} + \frac{x^5}{(n+4)\underline{|4}} + \&c$$

$$= e^x\left\{\frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c\right\}$$

Sol. $$L.H.S = \frac{1}{x^{n-1}}\left\{\frac{x^n}{n\underline{|0}} + \frac{x^{n+1}}{(n+1)\underline{|1}} + \frac{x^{n+2}}{(n+2)\underline{|2}} + \&c\right\}$$

$$= \frac{1}{x^{n-1}}\int x^{n-1}e^x\,dx$$

$$= \frac{e^x}{x^{n-1}}\left\{\int x^{n-1}dx - \iint x^{n-1}(dx)^2 + \iiint x^{n-1}(dx)^3 - \&c\right\}$$

$$= \frac{e^x}{x^{n-1}}\left\{\frac{x^n}{n} - \frac{x^{n+1}}{n(n+1)} + \frac{x^{n+2}}{n(n+1)(n+2)} - \&c\right\} = R.H.S.$$

or thus:—

$$\frac{1}{n+m} = \frac{1}{n} - \frac{m}{n(n+m)} = \frac{1}{n} - \frac{m}{n(n+1)} + \frac{m(m-1)}{n(n+1)(n+m)}$$

$$= \frac{1}{n} - \frac{m}{n(n+1)} + \frac{m(m-1)}{n(n+1)(n+2)} - \frac{m(m-1)(m-2)}{n(n+1)(n+2)(n+m)}$$

&c &c.

$$\therefore \frac{1}{n+m} = \frac{1}{n} - \frac{m}{n(n+1)} + \frac{m(m-1)}{n(n+1)(n+2)} - \&c$$

$$\therefore \frac{1}{(n+m)\underline{|m}} = \frac{1}{n\underline{|m}} - \frac{1}{n(n+1)\underline{|m-1}} + \frac{1}{n(n+1)(n+2)\underline{|m-2}} - \&c$$

But $\frac{1}{(n+m)\underline{|m}}$ is the coeff. of x^{m+1} in L.H.S and the other is that of x^{m+1} in R.H.S. $\therefore$ L.H.S = R.H.S.

Cor. $$\frac{x}{\underline{|1}} + \left(1+\tfrac{1}{2}\right)\frac{x^2}{\underline{|2}} + \left(1+\tfrac{1}{2}+\tfrac{1}{3}\right)\frac{x^3}{\underline{|3}} + \left(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}\right)\frac{x^4}{\underline{|4}} + \&c$$

$$= e^x\left\{\frac{x}{\underline{|1}} - \frac{x^2}{2\underline{|2}} + \frac{x^3}{3\underline{|3}} - \frac{x^4}{4\underline{|4}} + \&c\right\}$$

Sol. $$\frac{x}{n} + \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} + \&c = e^x\left\{\frac{x}{n\underline{|0}} - \frac{x^2}{(n+1)\underline{|1}} + \&c\right\}$$

Differentiating both sides with regards to n and then writing 1 for n we can get the result.

2. $\frac{x}{n+1} - \frac{x^2}{(n+1)(n+2)} + \frac{x^3}{(n+1)(n+2)(n+3)} - \&c$

$= \frac{f_0(x)}{n} - \frac{f_1(x)}{n^2} + \frac{f_2(x)}{n^3} - \frac{f_3(x)}{n^4} + \&c$

Sol. By III 1 we have,

$$e^x\left\{\frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c\right\}$$

$$= \frac{x}{n\underline{|0}} + \frac{x^2}{(n+1)\underline{|1}} + \frac{x^3}{(n+2)\underline{|2}} + \&c$$

Changing n to $n+1$, we have

$$e^x\left\{\frac{x}{n+1} - \frac{x^2}{(n+1)(n+2)} + \frac{x^3}{(n+1)(n+2)(n+3)} - \&c\right\}$$

$$= \frac{1}{n}\left\{\frac{x}{\underline{|0}} + \frac{x^2}{\underline{|1}} + \frac{x^3}{\underline{|2}} + \frac{x^4}{\underline{|3}} + \&c\right\}$$

$$- \frac{1}{n^2}\left\{\frac{1}{\underline{|0}}x + \frac{2}{\underline{|1}}x^2 + \frac{3}{\underline{|2}}x^3 + \frac{4}{\underline{|3}}x^4 + \&c\right\}$$

$$+ \frac{1}{n^3}\left\{\frac{1^2}{\underline{|0}}x + \frac{2^2}{\underline{|1}}x^2 + \frac{3^2}{\underline{|2}}x^3 + \frac{4^2}{\underline{|3}}x^4 + \&c\right\}$$

$- \&c = \frac{e^x}{n}f_0(x) - \frac{e^x}{n^2}f_1(x) + \frac{e^x}{n^3}f_2(x) - \&c$ by our supposition. $\therefore$ L.H.S = R.H.S.

N.B. An easier solution for III 1 is as follows —

Let $\phi(n) = \frac{x}{n\underline{|0}} + \frac{x^2}{(n+1)\underline{|1}} + \frac{x^3}{(n+2)\underline{|2}} + \&c$

Then $n\phi(n) = x + \frac{n}{n+1}\frac{x^2}{\underline{|1}} + \frac{n}{n+2}\frac{x^3}{\underline{|2}} + \&c$

and $x\phi(n+1) = \frac{x^2}{n+1} + \frac{x^3}{(n+2)\underline{|1}} + \&c$

$\therefore n\phi(n) + x\phi(n+1) = x + x^2 + \frac{x^3}{\underline{|2}} + \frac{x^4}{\underline{|3}} + \&c = xe^x$

$\therefore \phi(n) = e^x\frac{x}{n} - \frac{x}{n}\phi(n+1) = e^x\frac{x}{n} - e^x\frac{x^2}{n(n+1)} + \frac{x^2}{n(n+1)}\phi(n+2)$

&c &c.

$$\therefore \frac{x}{n\underline{|0}} + \frac{x^2}{(n+1)\underline{|1}} + \frac{x^3}{(n+2)\underline{|2}} + \frac{x^4}{(n+3)\underline{|3}} + \&c$$

$$= e^x\left\{\frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c\right\}$$

3. $e^{x(e^a-1)} = 1 + \frac{a}{\lfloor 1} f_0(x) + \frac{a^2}{\lfloor 2} f_1(x) + \frac{a^3}{\lfloor 3} f_2(x) + \&c$

Sol. $e^{xe^a} = 1 + xe^a + \frac{x^2}{\lfloor 2} e^{2a} + \frac{x^3}{\lfloor 3} e^{3a} + \&c.$

The coeff^t. of a^{n+1} is $\frac{1}{\lfloor n+1} \left\{ \frac{1^n}{\lfloor 0} x + \frac{2^n}{\lfloor 1} x^2 + \frac{3^n}{\lfloor 2} x^3 + \&c \right\}$

$= \frac{e^x}{\lfloor n+1} f_n(x).$

$\therefore e^{xe^a} = e^x \left\{ 1 + \frac{a}{\lfloor 1} f_0(x) + \frac{a^2}{\lfloor 2} f_1(x) + \frac{a^3}{\lfloor 3} f_2(x) + \&c \right\}.$

4. $f_n(x) = x \left\{ 1 + n f_0(x) + \frac{n(n-1)}{\lfloor 2} f_1(x) + \frac{n(n-1)(n-2)}{\lfloor 3} f_2(x) + \&c \right\}$

Sol. Differentiating both sides in III 3 with regards to a we have $x e^a e^{x(e^a-1)} = f_0(x) + \frac{a}{\lfloor 1} f_1(x) + \frac{a^2}{\lfloor 2} f_2(x) + \&c$

But $x e^a e^{x(e^a-1)} = x e^a \left\{ 1 + \frac{a}{\lfloor 1} f_0(x) + \frac{a^2}{\lfloor 2} f_1(x) + \&c \right\}$

Equating the coeff^{ts}. of a^n we get the results.

N.B. The above result may be written thus –

$$\begin{array}{ccccc} f_0(x), & f_1(x), & f_2(x), & f_3(x), & f_4(x) \\ & a_0 & b_0 & c_0 & d_0 \\ & & a_1 & b_1 & c_1 \\ & & & a_2 & b_2 \\ & & & & a_3 \end{array}$$

These are successive diff^s. a_n being equal to $x f_n(x)$.

5. If $f_n(x) = \phi_1(n) x + \phi_2(n) x^2 + \phi_3(n) x^3 + \dots + \phi_{n+1}(n) x^{n+1}$.

then $\frac{\phi_r(n)}{\lfloor 0} + \frac{\phi_{r-1}(n)}{\lfloor 1} + \frac{\phi_{r-2}(n)}{\lfloor 2} + \&c = \frac{r^n}{\lfloor r-1}$

Sol. $e^x f_n(x) = e^x \left\{ \phi_1(n) x + \phi_2(n) x^2 + \dots + \phi_{n+1}(n) x^{n+1} \right\}$

Again we have $e^x f_n(x) = \frac{1^n}{\lfloor 0} x + \frac{2^n}{\lfloor 1} x^2 + \frac{3^n}{\lfloor 2} x^3 + \&c$

By equating the coeff^{ts}. of x^r in both sides we can get the results.

$$\phi(0) + \frac{\phi(1)}{\lfloor 1} x + \frac{\phi(2)}{\lfloor 2} x^2 +$$

$$= e^x \left\{ \phi(x) + \frac{x}{2} \phi''(x) + \&c \right\} \text{ nearly}$$

6. $\phi_{n+1}(n) \lfloor n = (n+1)^n - n\cdot n^n + \frac{n(n-1)}{\lfloor 2}(n-1)^n - \frac{n(n-1)(n-2)}{\lfloor 3}(n-2)^n$

$+ \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}(n-3)^n - \&c$

Sol. $f_n(x) = \phi_1(n)x + \phi_2(n)x^2 + \phi_3(n)x^3 + \&c$

$= e^{-x}\left\{\frac{1^n}{\lfloor 0}x + \frac{2^n}{\lfloor 1}x^2 + \frac{3^n}{\lfloor 2}x^3 + \&c\right\}$

By equating the coeff^{ts} of x^{n+1} we can get the result.

7. $\phi_n(n+1) = n\phi_n(n) + \phi_{n-1}(n)$

Sol. $\phi_n(n+1) = \frac{1}{\lfloor n-1}\left\{n^{n+1} - (n-1)(n-1)^{n+1} + \frac{(n-1)(n-2)}{\lfloor 2}(n-2)^{n+1} - \&c\right\}$

$\therefore \phi_n(n+1) - \phi_{n-1}(n) = \frac{1}{\lfloor n-1}\left\{n\cdot n^n - n(n-1)(n-1)^n + \frac{n(n-1)(n-2)}{\lfloor 2}(n-2)^n - \&c\right\}$

$= n\phi_n(n)$. $\therefore \phi_n(n+1) = n\phi_n(n) + \phi_{n-1}(n)$. or in words thus

Write under each term the product of the coeff^t. and the index of x of that term together with the coeff^t. of the preceding one.

$f_0(x) = x$

$f_1(x) = x + x^2$

$f_2(x) = x + 3x^2 + x^3$

$f_3(x) = x + 7x^2 + 6x^3 + x^4$

$f_4(x) = x + 15x^2 + 25x^3 + 10x^4 + x^5$

$f_5(x) = x + 31x^2 + 90x^3 + 65x^4 + 15x^5 + x^6$

$f_6(x) = x + 63x^2 + 301x^3 + 350x^4 + 140x^5 + 21x^6 + x^7$

Ex 1. If $\frac{a_1}{n+1} - \frac{a_2}{(n+1)(n+2)} + \frac{a_3}{(n+1)(n+2)(n+3)} - \&c$

$= \frac{F(0)}{n} - \frac{F(1)}{n^2} + \frac{F(2)}{n^3} - \frac{F(3)}{n^4} + \&c$

Show that $F(n) = \phi_1(n)a_1 + \phi_2(n)a_2 + \phi_3(n)a_3 + \&c$

2. Show that $\phi_{n+1}(n)$ is the coeff^t. of $\frac{x^n}{\lfloor n}$ in $\frac{e^x}{\lfloor n}(e^x-1)^n$

Sol. From III 6 we have $\phi_{r+1}(n)\lfloor r$

$= (r+1)^n - \frac{r}{\lfloor 1} r^n + \frac{r(r-1)}{\lfloor 2}(r-1)^n - \&c$

$=$ the coeff^t. of $\frac{x^n}{\lfloor n}$ in $\left\{ e^{x(r+1)} - \frac{r}{\lfloor 1} e^{xr} + \frac{r(r-1)}{\lfloor 2} e^{x(r-1)} - \&c \right\}$

$=$ the coeff^t. of $\frac{x^n}{\lfloor n}$ in $e^x(e^x-1)^r$.

3. $\frac{d f_{n-1}(x)}{dx} = n f_{n-2}(x) + \frac{n(n-1)}{\lfloor 2} f_{n-3}(x) + \frac{n(n-1)(n-2)}{\lfloor 3} f_{n-4}(x) + \&c$

Sol. Differentiating both sides in III 3 with regards to x and then differentiating the result with regards to a and then equating the coeff^{ts}. as in III 4. we can get the result.

4. $\frac{1}{2} f_n(x) + \int f_n(x)\,dx = \frac{f_{n+1}(x)}{n+1} + \frac{B_2}{\lfloor 2} n f_{n-1}(x)$
$- \frac{B_4}{\lfloor 4} n(n-1)(n-2) f_{n-3}(x) + \frac{B_6}{\lfloor 6} n(n-1)(n-2)(n-3)(n-4) f_{n-5}(x) - \&c$

Sol. Integrating both sides in III 3. with regards to x we have $\frac{1}{e^a-1}\left\{1 + \frac{a}{\lfloor 1} f_0(x) + \frac{a^2}{\lfloor 2} f_1(x) + \frac{a^3}{\lfloor 3} f_2(x) + \&c\right\}$

$= \frac{1}{e^a-1} + x + \frac{a}{\lfloor 1}\int f_0(x)\,dx + \frac{a^2}{\lfloor 2}\int f_1(x)\,dx + \frac{a^3}{\lfloor 3}\int f_2(x)\,dx + \&c$

Equating the coeff^{ts}. of a^{n+1} in both sides we can get the results.

5. If $\frac{1^n}{\lfloor 0} + \frac{2^n}{\lfloor 1} + \frac{3^n}{\lfloor 2} + \frac{4^n}{\lfloor 3} + \&c = eA_n$

Shew that $A_0 = 1$, $A_1 = 2$, $A_2 = 5$, $A_3 = 15$, $A_4 = 52$, $A_5 = 203$, $A_6 = 877$, $A_7 = 4140$, $A_8 = 21147$ &c

Sol. $2 = 1+1$, $5 = 1+2\cdot1+2$, $15 = 1+3\cdot1+3\cdot2+5$, $52 = 1+4\cdot1+6\cdot2+4\cdot5+15$, $203 = 1+5\cdot1+10\cdot2+10\cdot5+5\cdot15+52$ &c

6. If $-\frac{1^n}{\lfloor 0} + \frac{2^n}{\lfloor 1} - \frac{3^n}{\lfloor 2} + \frac{4^n}{\lfloor 3} - \&c = \frac{A_n}{e}$, shew that

$A_0 = -1$, $A_1 = 0$, $A_2 = 1$, $A_3 = 1$, $A_4 = -2$, $A_5 = -9$, $A_6 = -9$, $A_7 = 50$

$A_8 = 267$ &c &c

N.B.

1	2	5	15	52	203	877
	1	3	10	37	151	674
		2	7	27	114	523
			5	20	87	409
				15	67	322
					52	255
						203

−1	0	1	1	−2	−9	−9	50
	1	1	0	−3	−7	0	59
		0	−1	−3	−4	7	59
			−1	−2	−1	11	52
				−1	1	12	41
					2	11	29
						9	18
							9

7 Shew that

i. $\frac{1^3}{\lfloor\underline{0}} + \frac{2^3}{\lfloor\underline{1}} + \frac{3^3}{\lfloor\underline{2}} + \frac{4^3}{\lfloor\underline{3}} + \&c = 3\left(\frac{1^2}{\lfloor\underline{0}} + \frac{2^2}{\lfloor\underline{1}} + \frac{3^2}{\lfloor\underline{2}} + \&c\right)$

ii. $\frac{1^2(1^3+1)}{\lfloor\underline{0}} + \frac{2^2(2^3+1)}{\lfloor\underline{1}} + \frac{3^2(3^3+1)}{\lfloor\underline{2}} + \&c = 4\left(\frac{1^4}{\lfloor\underline{0}} + \frac{2^4}{\lfloor\underline{1}} + \frac{3^4}{\lfloor\underline{2}} + \&c\right)$

iii. $\frac{1^3}{\lfloor\underline{0}} - \frac{2^3}{\lfloor\underline{1}} + \frac{3^3}{\lfloor\underline{2}} - \frac{4^3}{\lfloor\underline{3}} + \&c = \frac{1^2}{\lfloor\underline{0}} - \frac{2^2}{\lfloor\underline{1}} + \frac{3^2}{\lfloor\underline{2}} - \frac{4^2}{\lfloor\underline{3}} + \&c$

iv. $\frac{1^6}{\lfloor\underline{0}} - \frac{2^6}{\lfloor\underline{1}} + \frac{3^6}{\lfloor\underline{2}} - \frac{4^6}{\lfloor\underline{3}} + \&c = \frac{1^5}{\lfloor\underline{0}} - \frac{2^5}{\lfloor\underline{1}} + \frac{3^5}{\lfloor\underline{2}} - \frac{4^5}{\lfloor\underline{3}} + \&c$

v. $\frac{1^3(1^2+1)(1^3+1)}{\lfloor\underline{0}} - \frac{2^3(2^2+1)(2^3+1)}{\lfloor\underline{1}} + \frac{3^3(3^2+1)(3^3+1)}{\lfloor\underline{2}} - \&c$

$= 5\left(\frac{1^7}{\lfloor\underline{0}} - \frac{2^7}{\lfloor\underline{1}} + \frac{3^7}{\lfloor\underline{2}} - \frac{4^7}{\lfloor\underline{3}} + \frac{5^7}{\lfloor\underline{4}} - \&c\right).$

8. If $\frac{x}{n+a+b} - \frac{bx^2}{(n+a+b)(n+a+2b)} + \frac{b^2x^3}{(n+a+b)(n+a+2b)(n+a+3b)}$

$- \&c = \frac{F_0(x)}{n} - \frac{F_1(x)}{n^2} + \frac{F_2(x)}{n^3} - \frac{F_3(x)}{n^4} + \&c$, then

$(a+b)^n\frac{x}{\lfloor\underline{0}} + (a+2b)^n\frac{x^2}{\lfloor\underline{1}} + (a+3b)^n\frac{x^3}{\lfloor\underline{2}} + \&c = e^x F_n(x).$

Cor 1. $F_0(x) + yF_1(x) + \frac{y^2}{\lfloor\underline{2}}F_2(x) + \frac{y^3}{\lfloor\underline{3}}F_3(x) + \&c = x\,e^{y(a+b)}\,e^{x(e^{by}-1)}$

Cor 2. $F_{n+1}(x) - (a+b)F_n(x) = bx\left\{F_n(x) + \frac{n}{\lfloor\underline{1}}F_{n-1}(x)\,b + \frac{n(n-1)}{\lfloor\underline{2}}F_{n-2}(x)\,b^2 \right.$

$\left. + \&c\right\}$

Cor 3. If $F_n(x) = \phi_1(n)x + \phi_2(n)x^2 + \phi_3(n)x^3 + \&c$, then

$\frac{\phi_n(x)}{\lfloor\underline{0}} + \frac{\phi_{n-1}(x)}{\lfloor\underline{1}} + \frac{\phi_{n-2}(x)}{\lfloor\underline{2}} + \&c = \frac{(a+nb)^n}{\lfloor\underline{n-1}}.$

N.B. If $F_{n+1}(x) - (a+b)F_n(x) = \psi_n(x)$, then

$\psi_0(x)$	$\psi_1(x)$	$\psi_2(x)$	$\psi_3(x)$	$\psi_4(x)$	$\psi_5(x)$
	a_1	b_1	c_1	d_1	e_1
		a_2	b_2	c_2	d_2
			a_3	b_3	c_3
				a_4	b_4
					a_5

These are successive diff^ces, b times the previous term being subtracted from each term and a_n being equal to $b \times F_n(x)$.

Cor 4. $\phi_n(n)\lfloor n-1 = (a+nb)^n - \frac{n-1}{\lfloor 1}(a+\overline{n-1}\,b)^n + \frac{(n-1)(n-2)}{\lfloor 2}(a+\overline{n-2}\,b)^n$
$- \frac{(n-1)(n-2)(n-3)}{\lfloor 3}(a+\overline{n-3}\,b)^n + \&c$

Cor 5. $\phi_n(n+1) = (a+nb)\,\phi_n(n) + b\,\phi_{n-1}(n)$ or in words thus — Write under each term the product of $a+nb$, n being the index of x, and the coeff^t of x of that term together with b times the coeff^t of the preceding one.

$F_0(x) = x$

$F_1(x) = (a+b)x + bx^2$

$F_2(x) = (a+b)^2x + b(2a+3b)x^2 + b^2x^3$

$F_3(x) = (a+b)^3x + b\{3(a+b)(a+2b) + b^2\}x^2 + 3b^2(a+2b)x^3 + b^3x^4$

$F_4(x) = (a+b)^4x + b\{2(a+b)(a+2b) + b^2\}(2a+3b)x^2 + b^2\{6(a+2b)^2 + b^2\}$
$\times x^3 + 2b^3(2a+5b)x^4 + b^4x^5.$

Ex. 1. Show that $\phi_{n+1}(n)$ is the coeff^t of $\frac{x^n}{\lfloor n}$ in $\frac{e^{x(a+b)}}{\lfloor n}(e^{bx}-1)^n$.

2 Show that i. $\frac{1^3+1^2}{\lfloor 0} - \frac{3^3+3^2}{\lfloor 1} + \frac{5^3+5^2}{\lfloor 2} - \&c = 0$

ii. $\frac{1^4}{\lfloor 0} + \frac{2^4}{\lfloor 1} + \frac{3^4}{\lfloor 2} + \frac{4^4}{\lfloor 3} + \&c = 4\left(\frac{1^2}{\lfloor 0} + \frac{3^2}{\lfloor 1} + \frac{5^2}{\lfloor 2} + \&c\right)$

iii. $\frac{1^7+1^6}{\lfloor 0} - \frac{2^7+2^6}{\lfloor 1} + \frac{3^7+3^6}{\lfloor 2} - \&c = \frac{1^4}{\lfloor 0} - \frac{3^4}{\lfloor 1} + \frac{5^4}{\lfloor 2} - \frac{7^4}{\lfloor 3} + \&c$ -

iv. $1^4 - \frac{3^4}{1\lfloor 1} + \frac{5^4}{2\lfloor 2} - \frac{7^4}{3\lfloor 3} + \&c = \left(1 - \frac{1}{1\lfloor 1} + \frac{1}{2\lfloor 2} - \frac{1}{3\lfloor 3} + \&c\right) - 4.$

CHAPTER IV

1. If $n^r + \frac{(n+1)^{r+1}}{a\lfloor 1} + \frac{(n+2)^{r+2}}{a^2\lfloor 2} + \frac{(n+3)^{r+3}}{a^3\lfloor 3} + \&c = F_r(n)$, then

$$F_{r+1}(n) = n\,F_r(n) + \frac{1}{a}F_{r+1}(n+1).$$

Sol. $F_{r+1}(n) = n^{r+1} + \frac{(n+1)^{r+2}}{a\lfloor 1} + \frac{(n+2)^{r+3}}{a^2\lfloor 2} + \&c$

$= n\left\{n^r + \frac{(n+1)^{r+1}}{a\lfloor 1} + \frac{(n+2)^{r+2}}{a^2\lfloor 2} + \&c\right\} + \frac{1}{a}\left\{(n+1)^{r+1} + \frac{(n+2)^{r+2}}{a\lfloor 1} + \&c\right\}$

$= n\,F_r(n) + \frac{1}{a}F_{r+1}(n+1)$.

We see from this identity that if we are able to find the sum for one value of r we can sum up the series for all values of r.

2. If $x = a\log_e x$, then $\frac{x^n}{n} = n^{-1} + \frac{(n+1)^0}{a\lfloor 1} + \frac{(n+2)^1}{a^2\lfloor 2} + \frac{(n+3)^2}{a^3\lfloor 3} + \&c$

Sol. Suppose $f(n) = 1 + \frac{n}{a\lfloor 1} + \frac{n(n+2)}{a^2\lfloor 2} + \frac{n(n+3)^2}{a^3\lfloor 3} + \&c$

If we multiply $f(n)$ by $f(m)$ we get $f(m+n)$.

$\therefore f(n) = \{f(1)\}^n$. Let $f(1) = x$ then $x^n = f(n)$.

$\frac{f(n)-1}{n}$, when $n=0$, $= \frac{1}{a} + \frac{2}{a^2\lfloor 2} + \frac{3^2}{a^3\lfloor 3} + \&c$

$= \frac{1}{a}\left(1 + \frac{1}{a\lfloor 1} + \frac{3}{a^2\lfloor 2} + \frac{4^2}{a^3\lfloor 3} + \&c\right) = \frac{1}{a}f(1) = \frac{x}{a}$

i.e. $\frac{x^n-1}{n}$ when $n=0$, $= \frac{x}{a}$ or $\log_e x = \frac{x}{a}$ or $x = a\log_e x$.

N.B. The minimum value of $\frac{x}{\log_e x}$ is e. If $a=e$ $f(n) = e^n$.

If $a > e$ it is convergent but if $a < e$ it is divergent.

Cor. $e^x = 1 + \frac{x}{e^n} + \frac{x(x+2n)}{e^{2n}\lfloor 2} + \frac{x(x+3n)^2}{e^{3n}\lfloor 3} + \frac{x(x+4n)^3}{e^{4n}\lfloor 4} + \&c$

Sol. Write e^x for x and $\frac{n}{x}$ for n in IV 2.

N.B. In a similar manner we can prove that if $aqx^p - x^q + 1 = 0$, then

$$x^n = 1 + \frac{n}{\lfloor 1}a + \frac{n(n+2p-q)}{\lfloor 2}a^2 + \frac{n(n+3p-q)(n+3p-2q)}{\lfloor 3}a^3$$
$$+ \frac{n(n+4p-q)(n+4p-2q)(n+4p-3q)}{\lfloor 4}a^4 + \&c.$$

Ex. 1. Shew that

$$2^m = 1 + \frac{m}{2^n} + \frac{m(m+2n-1)}{2^{2n}\lfloor 2} + \frac{m(m+3n-1)(m+3n-2)}{2^{3n}\lfloor 3} + \&c$$

2. $$\left(\frac{2}{1+\sqrt{1-4x}}\right)^n = 1 + nx + \frac{n(n+3)}{\lfloor 2} x^2 + \frac{n(n+4)(n+5)}{\lfloor 3} x^3 + \&c$$

3. Expand x in terms of a in each of the following

i $x^a = e^{\pm x}$; Sol $a \log_e x = \pm x$.

ii. $x^a = a^{\pm x}$; Sol. $a \log_e x = \pm x \log_e a$ $\therefore \frac{x}{\log_e x} = \pm \frac{a}{\log_e a}$

iii. $x = a e^{\pm x}$; Sol. Let $x = \log_e y$, then $\log_e y = a y^{\pm 1}$

iv. $x = a^{\pm x}$; Sol. Let $x \log_e a = \log_e y$, then $\log_e y = y^{\pm 1} \log_e a$.

v. $x^{\pm x} = a$; Sol. Let $x = \frac{1}{y}$ then $y = a^{\mp y}$

vi. $x e^{\pm x} = a$; Sol. Let $x = \log_e y$ then $\log_e y = a y^{\pm 1}$

vii. $e^x \pm x = a$; Sol. Let $x = \log_e \log_e y$ then $e^a = y(\log_e y)^{\pm 1}$

viii. $x \pm \log_e x = a$; Sol. Let $x = \log_e y$ then $e^a = y(\log_e y)^{\pm 1}$

4. Shew how to find the values of the following for special values of x

i. $x^{x^{x^{\&c}}}$; Sol. let its value be equal to v. then $x^v = v$.

ii. $x \pm e^{x \pm e^{x \pm \&c}}$ Sol. $x \pm e^v = v$.

iii. $\log_e\left\{x \log_e\left[x \log_e(x \ldots \text{ad inf.})\right]\right\}$

iv. $\pm \log_e\left\{x \pm \log_e\left[x \pm \log_e(x \pm \&c)\right]\right\}$

3. If we write $x^n \phi_n(n)$ for $F_n(x)$, we see that

$$\phi_{n+1}(n) - \log_e x \; \phi(n+1, n-1) = n \phi(n, x).$$

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Case I If r is positive,

Let $$\phi_r(n) = \frac{\psi_1(r,n)}{(1-\log_e x)^{r+1}} + \frac{\psi_2(r,n)}{(1-\log_e x)^{r+2}} + \cdots + \frac{\psi_{r+1}(r,n)}{(1-\log_e x)^{2r+1}}.$$

Then by IV.1 we see that

$$n\,\psi_t(r,n) + \psi_{t-1}(r+1, n+1) = \psi_t(r+1, n+1) + \psi_{t-1}(r+1, n).$$

Case II If r is negative the terms in R.H.S continue as far as the term independent of $(1-\log_e x)$.

N.B $F_r(n)$ is convergent when $a > e$ or $\not> e$ according as r is positive or not.

Cor. 1. $\psi_1(r,n) + \psi_2(r,n) + \psi_3(r,n) + \&c$ as far as the terms cease to continue in $\phi_r(n) = n^r$.

Sol. L.H.S $= \phi_r(n)$ when $x = 1$ i.e $F_r(n)$ when $x = 1$ i.e $F_r(n)$ when $a = \infty$ i.e n^r.

Cor 2. $$e^x = (1-n)\left\{1 + \frac{x+n}{e^n \underline{|1}} + \frac{(x+2n)^2}{e^{2n}\underline{|2}} + \frac{(x+3n)^3}{e^{3n}\underline{|3}} + \&c\right\}$$

N.B. $$\phi(-1, n) = \frac{1}{n}$$

$$\phi(-2, n) = \frac{1-\log_e x}{n(n+1)} + \frac{1}{n^2(n+1)}$$

$$\phi(-3, n) = \frac{(1-\log_e x)^2}{n(n+1)(n+2)} + \frac{(3n+2)(1-\log_e x)}{n^2(n+1)^2(n+2)} + \frac{3n+2}{n^3(n+1)^2(n+2)}$$

$$\phi(0, n) = \frac{1}{1-\log_e x}$$

$$\phi(1, n) = \frac{n-1}{(1-\log_e x)^2} + \frac{1}{(1-\log_e x)^3}$$

$$\phi(2, n) = \frac{(n-1)(n-2)}{(1-\log_e x)^3} + \frac{(n-1)(n-2)\left(\frac{1}{n-2} + \frac{2}{n-1}\right)}{(1-\log_e x)^4} + \frac{1.3}{(1-\log_e x)^5}$$

$$\phi(3, n) = \frac{(n-1)(n-2)(n-3)}{(1-\log_e x)^4} + \frac{(n-1)(n-2)(n-3)\left(\frac{1}{n-3} + \frac{2}{n-2} + \frac{3}{n-1}\right)}{(1-\log_e x)^5} + \frac{15n-35}{(1-\log_e x)^6} + \frac{1.3.5}{(1-\log_e x)^7}$$

2. To expand x^m in ascending powers of h when $x^x = a^a e^h$.

Let $\frac{x-a}{a} = \frac{A_1}{\lfloor 1}\cdot\frac{h}{a} - \frac{A_2}{\lfloor 2}\left(\frac{h}{a}\right)^2 + \frac{A_3}{\lfloor 3}\left(\frac{h}{a}\right)^3 - \&c$

and let $n = \frac{1}{1+\log_e a}$

then $A_r - n(r-2)A_{r-1} = n\Big\{ rA_1A_{r-1} + \frac{r(r-1)}{\lfloor 2}A_2A_{r-2} +$

$\frac{r(r-1)(r-2)}{\lfloor 3}A_3A_{r-3} + \&c\Big\}$ the last term being

$\frac{\lfloor r}{\lfloor \frac{r-1}{2}\,\lfloor \frac{r+1}{2}} A_{\frac{r-1}{2}}$ or $\frac{\lfloor r}{2\left(\lfloor \frac{r}{2}\right)^2} A_{\frac{r}{2}}$ according as r is odd or even.

$A_1 = n$

$A_2 = n^3$

$A_3 = 3n^5 + n^4$

$A_4 = 15n^7 + 10n^6 + 2n^5$

$A_5 = 105n^9 + 105n^8 + 40n^7 + 6n^6$

$A_6 = 945n^{11} + 1260n^{10} + 700n^9 + 196n^8 + 24n^7$

$A_7 = 10395n^{13} + 17325n^{12} + 12600n^{11} + 5068n^{10} + 1148n^9 + 120n^8$

Multiply the power and the coefft. write under it the sum of this product and $(r-1)$ times the coefft. of the preceding term where r is the suffix of A.

N.B. For $\frac{a}{x}$ take $r+1$ times the coefft.; for $\log_e\frac{x}{a}$ take r times the coefft. and generally for $\frac{x^m}{a^m}$ take $r-m$ times the coefft.

Ex. 1. Shew that the sum of the coeffts. of A_r is $(r-1)^{r-1}$

Sol. Put 1 for a. Then $x^x = e^h$. Let $x = \frac{1}{y}$

then $y^{\frac{1}{y}} = e^{-h}$ or $\frac{1}{y}\log_e y = -h$

∴ By applying IV 2 we have $\frac{1}{y} = x = 1 + h - \frac{1^1}{\lfloor 2}h^2 + \frac{2^2}{\lfloor 3}h^3 - \frac{3^3}{\lfloor 4}h^4 + \&c$. ∴ The sum of the coeffts. of $A_r = (r-1)^{r-1}$.

2. Expand x in ascending powers of h when $x/x = e^{h\,a/a}$.

Let $x = \frac{1}{y}$ then $y^y = e^{-h}\left(\frac{1}{a}\right)^a$.

CHAPTER V

Let $F_1(x) = e^x - 1$, $F_2(x) = e^{e^x-1}-1$,

$F_3(x) = e^{e^{e^x-1}-1}-1$, $F_4(x) = e^{e^{e^{e^x-1}-1}-1}-1$, $F_5(x) = e^{e^{e^{e^{e^x-1}-1}-1}-1}-1$ &c &c

Now let us try to find the expansion of $F_r(x)$

(1) In ascending powers of x

(2) In ascending powers of r

1. Let $e^{e^{\cdot^{\cdot^{\cdot^{e^x-1}}}}-1}-1$ (r times) $= F_r(x) = x\phi_1(r) + x^2\phi_2(r) + x^3\phi_3(r) + \&c$

$$= f_0(x) + r f_1(x) + r^2 f_2(x) + r^3 f_3(x) + \&c$$

then $\log_e\left[1 + \log_e\left\{1 + \log_e\left(1 + \dots + \log_e\overline{1+x}\right)\right\}\right]$ logarithms being taken r times

$$= F_{-r}(x) = x\phi_1(-r) + x^2\phi_2(-r) + x^3\phi_3(-r) + \&c$$

$$= f_0(x) - r f_1(x) + r^2 f_2(x) - r^3 f_2(x) + \&c$$

Sol. We have $e^{F_{r-1}(x)} - 1 = F_r(x)$. $\therefore F_{r-1}(x) = \log_e\{1 + F_r(x)\}$

$\therefore F_0(x) = x$ $\therefore F_{-1}(x) = \log_e(1+x)$, $\therefore F_{-2}(x) = \log_e\{1 + \log_e(1+x)\}$ &c &c

Cor. $F_0(x) = x$ and $f_0(x) = x$.

2. $\dfrac{dF_r(x)}{dx} \div \dfrac{dF_{r-1}(x)}{dx} = 1 + F_r(x)$.

Sol. $e^{F_{r-1}(x)} - 1 = F_r(x)$ $\therefore F_{r-1}(x) = \log_e\{1 + F_r(x)\}$

Differentiating both sides with regards to x we have

$$\frac{dF_r(x)}{dx} = \{1 + F_r(x)\}\frac{dF_{r-1}(x)}{dx}$$

Cor. $\dfrac{dF_r(x)}{dx} = \{1 + F_1(x)\}\{1 + F_2(x)\}\{1 + F_3(x)\}\cdots\cdots\{1 + F_r(x)\}$.

Sol. From V 2, we have $F_n'(x) = \{1 + F_n(x)\} F_{n-1}'(x)$

$= \{1 + F_n(x)\}\{1 + F_{n-1}(x)\} F_{n-2}'(x) = \{1 + F_n(x)\}\{1 + F_{n-1}(x)\}\{1 + F_{n-2}(x)\} F_{n-3}'(x)$

$= \&c\ \&c = \{1 + F_n(x)\}\{1 + F_{n-1}(x)\}\{1 + F_{n-2}(x)\} \cdots \{1 + F_1(x)\} F_0'(x)$.

But from V 1 Cor $F_0(x) = x \quad \therefore F_0'(x) = 1$

Cor 2. $n\{\phi_n(x) - \phi_n(n-1)\} = (n-1)\phi_1(x)\phi_{n-1}(n-1) + (n-2)\phi_2(x)\phi_{n-2}(n-1)$

$+ (n-3)\phi_3(x)\phi_{n-3}(n-1) + \&c.$

Sol. $F_{n-1}'(x)\{1 + F_n(x)\} = F_n'(x)$. Here equate the coeff[ts]. of x^{n+1}.

Cor 3. $\dfrac{d f_1(x)}{dx} = x - \frac{1}{2} f_1(x) + B_2 f_2(x) - B_4 f_4(x) + B_6 f_6(x) - \&c$

Sol. From V 2 cor 1 we have $F_n'(x) = \{1 + F_1(x)\}\{1 + F_2(x)\} \cdots \{1 + F_n(x)\}$

i.e. $1 + n\dfrac{d f_1(x)}{dx} + \&c = e^{F_0(x) + F_1(x) + \cdots + F_{n-1}(x)}$

$\therefore \log_e\left\{1 + n\dfrac{d f_1(x)}{dx} + \&c\right\} = F_0(x) + F_1(x) + F_2(x) + \&c$ to n terms.

$= \psi(x) + \int_0^n F_n(x)\,dn - \frac{1}{2}F_n(x) + \dfrac{B_2}{\lfloor 2}\dfrac{dF_n(x)}{dn} - \dfrac{B_4}{\lfloor 4}\dfrac{d^3F_n(x)}{dn^3} + \&c$

where $\psi(x)$ is a function of x independent of n.

Equating the coeff[ts]. of n, we have

$\dfrac{d f_1(x)}{dx} = x - \frac{1}{2}f_1(x) + B_2 f_2(x) - B_4 f_4(x) + B_6 f_6(x) - \&c.$

N. B. $\psi(x) = \displaystyle\int_0^x \frac{x - \frac{d f_1 x}{dx}}{f_1(x)}\,dx$

Sol. Since when $n = 0$, $\log_e\left\{1 + n\dfrac{d f_1(x)}{dx} + \&c\right\} = 0$

$\psi(x) = \dfrac{x}{2} - \dfrac{B_2}{2} f_1(x) + \dfrac{B_4}{4} f_3(x) - \dfrac{B_6}{6} f_5(x) + \&c$

$\psi'(x) = \dfrac{1}{2} - \dfrac{B_2}{2} f_1'(x) + \dfrac{B_4}{4} f_3'(x) - \dfrac{B_6}{6} f_5'(x) + \&c.$

$$\left\{1 + 2\left(\frac{\cos\theta}{\cosh\pi} + \frac{\cos 2\theta}{\cosh 2\pi} + \frac{\cos 3\theta}{\cosh 3\pi} + \&c.\right)\right\}^{-2}$$

$$+ \left\{1 + 2\left(\frac{\cosh\theta}{\cosh\pi} + \frac{\cosh 2\theta}{\cosh 2\pi} + \frac{\cosh 3\theta}{\cosh 3\pi} + \&c.\right)\right\}^{-2}$$

~~is constant~~ $= \frac{2}{\pi}\,(1-\frac{1}{4})^{4}$.

35. $$(\sqrt{5}-2)^{8}\left(\frac{\sqrt{4+\sqrt{7}} - \sqrt[4]{7}}{2}\right)^{36}(6-\sqrt{35})^{6}$$

$$\times\left(\sqrt{\frac{43+15\sqrt{7}+(8+3\sqrt{7})\sqrt{10\sqrt{7}}}{8}} \pm \sqrt{\frac{35+15\sqrt{7}+(8+3\sqrt{7})\sqrt{10\sqrt{7}}}{8}}\right)^{24}$$

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21. $$\left(\frac{\sqrt{4+\sqrt{7}} - \sqrt[4]{7}}{2}\right)^{24}\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{12}(2-\sqrt{3})^{4}$$

$$\times\left(\frac{\sqrt{3+\sqrt{7}} - \sqrt[4]{6\sqrt{7}}}{\sqrt{3+\sqrt{7}} + \sqrt[4]{6\sqrt{7}}}\right)^{12}$$

$\therefore \psi'(x) f_1(x) = \frac{1}{2} f_1(x) - \frac{B_2}{2} f_1(x) f_1'(x) + \frac{B_4}{4} f_1(x) f_3'(x) - \&c$

$= \frac{1}{2} f_1(x) - B_2 f_2(x) + B_4 f_4(x) - \&c$ by V 3.

$= x - f_1'(x)$ by V 2 Cor 3.

$\therefore \psi'(x) = \dfrac{x - f_1'(x)}{f_1(x)} \quad \therefore \psi(x) = \int_0^x \dfrac{x - f_1'(x)}{f_1(x)} dx$

3. $\dfrac{dF_r(x)}{dr} = f_1(x) \dfrac{dF_r(x)}{dx}$ and consequently $n f_n(x) = f_1(x) \dfrac{df_{n-1}(x)}{dx}$

Sol. In V 1. Write $F_k(x)$ for x; then $F_{r+k}(x) = F_r\{F_k(x)\}$.

But $F_{r+k}(x) = F_k(x) + r \dfrac{dF_k(x)}{dk} + \dfrac{r^2}{\lfloor 2} \dfrac{d^2F_k(x)}{dk^2} + \&c$

and $F_r\{F_k(x)\} = F_k(x) + r f_1\{F_k(x)\} + r^2 f_2\{F_k(x)\} + \&c.$

Equating the coeff^{ts}. of r we have $\dfrac{dF_k(x)}{dk} = f_1\{F_k(x)\}$

Let $F_k(x) = y$ and $F_k(y) = z$, then we have

$\dfrac{dy}{dk} = f_1(y) \quad \therefore \dfrac{dz}{dk} = f_1(y) \dfrac{dz}{dy}$.

$\therefore \dfrac{dF_k(y)}{dk} = f_1(y) \dfrac{dF_k(y)}{dy}$ or $\dfrac{dF_r(x)}{dr} = f_1(x) \dfrac{dF_r(x)}{dx}$

Equating the coeff^{ts} of r^{n+1} we have $n f_n(x) = f_1(x) f_{n-1}'(x)$.

Cor 1. If $f_n(x) = \left(\frac{x}{2}\right)^n \{\psi_1(n) x - \psi_2(n) x^2 + \psi_3(n) x^3 - \&c\}$, then

i. $n \psi_r(n) = n \psi_1(n-1) \psi_r(1) + (n+1) \psi_2(n-1) \psi_{r-1}(1) +$

$(n+2) \psi_3(n-1) \psi_{r-2}(1) + (n+3) \psi_4(n-1) \psi_{r-3}(1) + \&c.$

ii. $\phi_n(2r) = r^n \left\{ \dfrac{\psi_1(n-1)}{r} - \dfrac{\psi_2(n-2)}{r^2} + \dfrac{\psi_3(n-3)}{r^3} - \&c \right\}$

$n f_n(x) = f_1(x) f_{n-1}'(x)$ equate the coeff^{ts} of like powers

of x. $\phi_n(r)$ is the coeff^t. of x^n in $F_r(x)$ by I expans

$$\frac{1}{4n^2} + \frac{1}{n^2+(n+1)^2} + \frac{1}{n^2+(n+2)^2} + \frac{1}{n^2+(n+3)^2} + \cdots$$

$$= \frac{\pi}{4n} + \frac{1}{8\pi n^3} - \frac{\pi}{n}\cdot\frac{1}{e^{4\pi n} - 2e^{2\pi n}\cos 2\pi n + 1}$$

$$+ 4n\left\{\frac{1}{e^{2\pi}-1}\cdot\frac{1}{1^4+4n^4} + \frac{2}{e^{4\pi}-1}\cdot\frac{1}{2^4+4n^4} + \frac{3}{e^{6\pi}-1}\cdot\frac{1}{3^4+4n^4} + \&c\right\}$$

$$\frac{1}{2n^2} + \frac{1}{n^2+1^2} + \frac{1}{n^2+2^2} + \text{\&c ad inf.}$$

$$= \frac{\pi}{2n} + \frac{\pi}{n}\cdot\frac{1}{e^{2\pi n}-1}.$$

$$l\left\{\frac{1}{2(l^2+n^2)} + \frac{1}{l^2+(n+1)^2} + \frac{1}{l^2+(n+2)^2} + \&c\right\}$$

$$= \tan^{-1}\frac{l}{n} + \frac{B_2}{2}\cdot\frac{\sin\left(2\tan^{-1}\frac{l}{n}\right)}{l^2+n^2} - \frac{B_4}{4}\cdot\frac{\sin\left(4\tan^{-1}\frac{l}{n}\right)}{(l^2+n^2)^2}$$

$$+ \frac{B_6}{6}\cdot\frac{\sin\left(6\tan^{-1}\frac{l}{n}\right)}{(l^2+n^2)^3} - \&c$$

∴ Again find the coeff^t. of x^n by II expansion.
Equate the two results.

Cor 2. $(n+1)\,\psi_n(1) = \frac{1}{2}\psi_{n-1}(1) + \frac{B_2}{2}\psi_{n-2}(2) - \frac{B_4}{2^3}\psi_{n-4}(4)$
$+ \frac{B_6}{2^5}\psi_{n-6}(6) - \&c$

Sol. Equate the coeff^{ts}. of x^n in V 2 Cor 3.

4. $f_1(x) = (1+x)\,f_1\{\log_e(1+x)\}$

Sol. In VI write $\log_e(1+x)$ for x; then $F_{n-1}(x) = \log_e(1+x)$
$+ n f_1\{\log_e(1+x)\} + n^2 f_2\{\log_e(1+x)\} + \&c$

$\therefore e^{F_{n-1}(x)} = (1+x)\,e^{n f_1\{\log_e(1+x)\} + n^2 f_2\{\log_e(1+x)\} + \&c}$

But $e^{F_{n-1}(x)} = 1 + F_n(x) = 1 + x + n f_1(x) + n^2 f_2(x) + \&c$

$\therefore (1+x) + n(1+x)\,f_1\{\log_e(1+x)\} + \&c = (1+x) + n f_1(x) + \&c$

∴ Equating the coeff^{ts}. of n $f_1(x) = (1+x)\,f_1\{\log_e(1+x)\}$

5. i The sum of the coeff^{ts}. of the odd terms + the sum of the coeff^{ts}. of the even terms } in $\phi_n(a) = \frac{1}{n}$

ii The sum of the coeff^{ts} of the odd terms − the sum of the coeff^{ts}. of the even terms } in $\phi_n(n) = \frac{1}{[n}$

Sol. $F_1(x) = e^x - 1$ and $F_{-1}(x) = \log_e(1+x)$ Equate the coeff^{ts}.

iii. $f_0(x) = \frac{x^2}{2} - \frac{x^3}{12} + \frac{x^4}{48} - \frac{x^5}{180} + \frac{11x^6}{8640} - \frac{x^7}{6720} + \&c$

iv. $\phi_1(2n) = 1 \qquad \phi_3(2n) = n(n - \frac{1}{6})$
$\phi_2(2n) = n \qquad \phi_4(2n) = n(n - \frac{1}{3})(n - \frac{1}{4})$

$$(1+x)^n = 1 + nx(1+x)^{\frac{n-1}{2}} + \frac{n(n^2-1)}{4\lfloor\underline{3}}\,x^3(1+x)^{\frac{n-3}{2}} + \&c$$

$$+ \frac{n(n^2-1)(n^2-9)}{4^2\lfloor\underline{5}}\,x^5(1+x)^{\frac{n-5}{2}} + \&c$$

$$\left(\frac{1+\sqrt{1+4x}}{2}\right)^n = 1 + nx(1+x)^{\frac{n-3}{2}} + \frac{n(n-5)(n-7)}{4\lfloor\underline{3}}\,x^3(1+x)^{\frac{n-9}{2}}$$

$$+ \frac{n(n-7)(n-9)(n-11)(n-13)}{4^2\lfloor\underline{5}}\,x^5(1+x)^{\frac{n-15}{2}}$$

$$\frac{1+(1+x)^n}{2} = (1+x)^{\frac{n}{2}} + \frac{n^2}{4\lfloor\underline{2}}\cdot x^2(1+x)^{\frac{n-2}{2}}$$

$$+ \frac{n^2(n^2-2^2)}{4^2\lfloor\underline{4}}\,x^4(1+x)^{\frac{n-4}{2}} + \&c.$$

$$\frac{1}{2} + \frac{1}{2}\cdot\left(\frac{1+\sqrt{1+4x}}{2}\right)^n = (1+x)^{\frac{n}{2}} + \frac{n(n-4)}{4\lfloor\underline{2}}\,x^2(1+x)^{\frac{n-6}{2}}$$

$$+ \frac{n(n-6)(n-8)(n-10)}{4^2\lfloor\underline{4}}\,x^4(1+x)^{\frac{n-12}{2}} + \&c.$$

$\phi_1(2n) = 1$

$\phi_2(2n) = n$

$\phi_3(2n) = n^2 - \frac{n}{6}$

$\phi_4(2n) = n^3 - \frac{5n^2}{12} + \frac{n}{24}$

$\phi_5(2n) = n^4 - \frac{13n^3}{18} + \frac{n^2}{6} - \frac{n}{90}$

$\phi_6(2n) = n^5 - \frac{77n^4}{72} + \frac{89n^3}{216} - \frac{91n^2}{1440} + \frac{11n}{4320}$

$\phi_7(2n) = n^6 - \frac{29n^5}{20} + \frac{175n^4}{216} - \frac{149n^3}{720} + \frac{91n^2}{4320} - \frac{n}{3360}$

6. $\psi_1(n) = 1$

$\psi_2(n-1) = \frac{n}{6}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}\right)$

$\psi_3(n-2) = \frac{n(n-1)}{72}\left\{\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}\right)^2 - \left(\frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}\right) - \frac{1}{n} + \frac{1}{2}\right.$

Cor. If $\frac{x}{1-nx} = y$ and $(1-nx) = z$, then

$F_{2n}(x) = y + \frac{y^2}{6}\log_e z + \frac{y^3}{72}\left\{(\log_e z)^2 + (1-\log_e z)^2 - z\right\} + \&c$

Sol. Apply the above results in V 1.

Ex. Shew that $f_1(x) f_1''(x) = f_1(x) - f_2(x) + 3B_2 f_3(x) - 5B_4 f_5(x) + \&c$

Sol. From V 2 Cor 3 we have $f_1'(x) = x - \frac{1}{2} f_1(x) + B_2 f_2(x)$
$- B_4 f_4(x) + B_6 f_6(x) - \&c$. Differentiating both sides and multiplying the results by $f_1(x)$ we have
$f_1(x) f_1''(x) = f_1(x) - \frac{1}{2} f_1(x) f_1'(x) + B_2 f_1(x) f_2'(x) - B_4 f_1(x) f_4'(x) + \&c$
$= f_1(x) - \frac{1}{2} f_1(x) f_1'(x) + 3B_2 f_3(x) - 5B_4 f_5(x) + \&c$ by V 3.

7. i. $\Sigma 1 + \Sigma\frac{1}{2} + \Sigma\frac{1}{3} + \dots + \Sigma\frac{1}{x} = (x+1)\Sigma\frac{1}{x} - x$.

ii. $\left(\Sigma\frac{1}{1}\right)^2 + \left(\Sigma\frac{1}{2}\right)^2 + \left(\Sigma\frac{1}{3}\right)^2 + \dots + \left(\Sigma\frac{1}{x}\right)^2 = (x+1)\left(\Sigma\frac{1}{x}\right)^2 - (2x+1)\Sigma\frac{1}{x} + 2x$.

iii. $\left(\Sigma\frac{1}{1}\right)^3 + \left(\Sigma\frac{1}{2}\right)^3 + \left(\Sigma\frac{1}{3}\right)^3 + \dots + \left(\Sigma\frac{1}{x}\right)^3 = (x+1)\left(\Sigma\frac{1}{x}\right)^3 - 3\left(x+\frac{1}{2}\right)\left(\Sigma\frac{1}{x}\right)^2$
$+ \frac{3}{2}(2x+1)\Sigma\frac{1}{x} - 6x + \frac{1}{2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{x^2}\right)$

$$C + \log n + \frac{n}{1|\underline{1}} + \frac{n^2}{2|\underline{2}} + \frac{n^3}{3|\underline{3}} + \&c$$

$$= e^n\left(\frac{1}{n} + \frac{|\underline{1}}{n^2} + \frac{|\underline{2}}{n^3} + \cdots\cdots + \frac{|\underline{n-1}}{n^n}\theta\right)$$

where $\theta = \frac{2}{3} + \frac{4}{135n} + \frac{8}{27.105n^2} - \&c$

If $\displaystyle\int_0^{n(1-h)} \frac{1-e^{-x}}{x}\,dx = C + \log_e n$

then $h(e^n - 1) + \frac{h^2}{2}(e^n - 1 - n) + \frac{h^3}{3}\left(e^n - 1 - n - \frac{n^2}{|\underline{2}}\right) + \&c$

$$= \frac{1}{n} - \frac{|\underline{1}}{n^2} + \frac{|\underline{2}}{n^3} - \frac{|\underline{3}}{n^4} + \&c$$

CHAPTER VI

1. If $f(x+h) - f(x) = h\phi'(x)$ then

$$f(x) = \phi(x) - \frac{h}{2}\phi'(x) + \frac{B_2}{\underline{|2}} h^2 \phi''(x) - \frac{B_4}{\underline{|4}} h^4 \phi^{iv}(x) + \frac{B_6}{\underline{|6}} h^6 \phi^{vi}(x) - \&c$$

2. If $f(x+h) + f(x) = h\phi'(x)$ then

$$f(x) = \frac{h}{2}\phi'(x) - (2^2-1) B_2 \frac{h^2}{\underline{|2}} \phi''(x) + (2^4-1) B_4 \frac{h^4}{\underline{|4}} \phi^{iv}(x) - \&c$$

Sol. If we write e^x for $\phi(x)$, we see that the coeff^{ts} in R.H. of VI 1. are the coeff^{ts} in the expansion of $\frac{h}{e^h - 1}$

Again, if we write e^x for $\phi(x)$ in VI 2 we see the coeff^{ts} in VI 2 are the coeff^{ts} in the expansion of $\frac{h}{e^h+1}$ or

$$\frac{h}{e^h - 1} - \frac{2h}{e^{2h} - 1}.$$

3. Let $F_n(x) = \phi(x) - \frac{n-1}{n+1}\left\{\phi(x+h) + \phi(x-h)\right\} +$

$$\frac{(n-1)(n-3)}{(n+1)(n+3)}\left\{\phi(x+2h) + \phi(x-2h)\right\} - \frac{(n-1)(n-3)(n-5)}{(n+1)(n+3)(n+5)} \times$$

$\left\{\phi(x+3h) + \phi(x-3h)\right\} + \&c$, then

i. If $f(x+h) - f(x-h) = 2h\phi'(x)$, then

$$f(x) = \frac{F_1(x)}{1} + \frac{F_3(x)}{3} + \frac{F_5(x)}{5} + \frac{F_7(x)}{7} + \&c$$

ii. If $f(x+h) + f(x-h) = 2\phi(x)$, then

$$f(x) = F_1(x) + \frac{1}{\underline{|2}} F_3(x) + \frac{1.3}{\underline{|2}} F_5(x) + \frac{1.3.5}{\underline{|3}} F_7(x) + \&c$$

4. If $f(x+h) + K f(x) = \phi(x)$

let $f(x) = \frac{\phi(x)\psi_0(K)}{K+1} - \frac{h}{\underline{|1}} \cdot \frac{\phi'(x)\psi_1(K)}{(K+1)^2} + \frac{h^2}{\underline{|2}} \frac{\phi''(x)\psi_2(K)}{(K+1)^3}$ &c

then $\frac{1}{e^x + K} = \frac{\psi_0(K)}{K+1} - \frac{x}{\underline{|1}} \frac{\psi_1(K)}{(K+1)^2} + \frac{x^2}{\underline{|2}} \frac{\psi_2(K)}{(K+1)^3} - \&c$

Sol. Let $\phi(x) = e^x$, then $f(x) = \frac{e^x}{e^h + K}$

$$\int \phi(x) e^{-nx} dx = -e^{-nx}\left\{\frac{\phi(x)}{n} + \frac{\phi'(x)}{n^2} + \frac{\phi''(x)}{n^3} + \&c\right\}$$

$$\int \phi(x) \cos nx\, dx = \sin nx\left\{\frac{\phi(x)}{n} - \frac{\phi''(x)}{n^3} + \&c\right\}$$
$$+ \cos nx\left\{\frac{\phi'(x)}{n^2} - \frac{\phi'''(x)}{n^4} + \&c\right\}$$

$$\int_x^\infty e^{-x^2} \cos 2nx\, dx$$
$$= e^{-x^2}\left\{\frac{\cos(2nx+\theta)}{2r} - \frac{1.\cos(2nx+3\theta)}{2^2 r^3} + \frac{1.3\cos(2nx+5\theta)}{2^3 r^5}\right.$$
$$\left. - \frac{1.3.5\cos(2nx+7\theta)}{2^4 r^7} + \&c\right\}$$

where $\tan\theta = \frac{n}{x}$ & $r = \sqrt{n^2+x^2}$

$$\int_0^\infty e^{-x^2}\left\{e^{2nx}\phi(x) + e^{-2nx}\phi(-x)\right\} dx$$
$$= \sqrt{\pi}\, e^{n^2}\left\{\phi(n) + \frac{\phi''(n)}{4} + \frac{\phi^{IV}(n)}{4.8} + \frac{\phi^{VI}(n)}{4.8.12} + \&c\right\}$$
$$= \int_0^\infty e^{n^2-x^2}\left\{\phi(n+x) + \phi(n-x)\right\}$$

$$\int_0^\infty e^{-\frac{1}{2}x^2}\left\{A_1 - \frac{x^2}{\underline{|2}} A_3 + \frac{x^4}{\underline{|2}} A_5 - \&c\right\} dx$$
$$= \sqrt{\frac{\pi}{2}}\left\{A_0 - \frac{2}{\underline{|1}} A_1 + \frac{2^2}{\underline{|2}} A_2 - \frac{2^3}{\underline{|3}} A_3 + \&c\right\}$$

Cor. 1. $1^n - 2^n k + 3^n k^2 - 4^n k^3 + 5^n k^4 - \&c = \dfrac{\psi_n(k)}{(k+1)^{n+1}}$

Sol. $\dfrac{1}{e^x + k} = e^{-x} - k e^{-2x} + k^2 e^{-3x} - k^3 e^{-4x} + \&c$

$= \dfrac{\psi_0(k)}{k+1} - \dfrac{x}{\lfloor 1} \dfrac{\psi_1(k)}{(k+1)^2} + \dfrac{x^2}{\lfloor 2} \dfrac{\psi_2(k)}{(k+1)^3} - \&c$ by VI 4.

Equate the coeff^{ts}. of x^n.

Cor 2. $\psi_0(k) - n \dfrac{\psi_1(k)}{k+1} + \dfrac{n(n-1)}{\lfloor 2} \dfrac{\psi_2(k)}{(k+1)^2} \dots\dots + (-1)^n \dfrac{\psi_n(k)}{(k+1)^n}$

$= (-1)^{n+1} \dfrac{k\, \psi_n(k)}{(k+1)^n}$

Sol. Multiply both sides in VI 4 by $e^x + k$, then we see that the coeff^t. of $x^n = 0$

Cor 3. If $\psi_n(k) = F_1(n) - F_2(n)k + F_3(n)k^2 - F_4(n)k^3 + \&c$

$\dots\dots + (-1)^{n+1} F_n(k) k^{n-1}$ then i. $F_{n-r}(n) = F_{r+1}(n)$.

ii. $F_r(n-1) + n F_{r-1}(n-1) + \dfrac{n(n+1)}{\lfloor 2} F_{r-2}(n-1) + \dots + \dfrac{\lfloor n+r-1}{\lfloor r \lfloor n-1} F_0(n-1)$

$= r^n$ Sol. Equate the coeff^{ts}. of k^{r-1} in VI 4. Cor 1

Cor 4. $F_r(n-1) = r^{n-1} - n(r-1)^{n-1} + \dfrac{n(n-1)}{\lfloor 2}(r-2)^{n-1} - \&c$ to $r+1$ terms

Sol. Multiply both sides in VI 4 Cor. by $(1+k)^{n+1}$ and then equate the coeff^{ts}. of k^{r-1}

Cor 5. $\psi_n(-1) = \lfloor n$, $\psi_n(1) = 2^{n+1}(2^{n+1}-1) \dfrac{B_{n+1}}{n+1} \cdot \sin\dfrac{\pi n}{2}$. $\psi_0(k) = 1$

$\psi_1(k) = 1$

$\psi_2(k) = 1 - k$

$\psi_3(k) = 1 - 4k + k^2$

$\psi_4(k) = 1 - 11k + 11k^2 - k^3$

$\psi_5(k) = 1 - 26k + 66k^2 - 26k^3 + k^4$

$\psi_6(k) = 1 - 57k + 302k^2 - 302k^3 + 57k^4 - k^5$

$\psi_7(k) = 1 - 120k + 1191k^2 - 2416k^3 + 1191k^4 - 120k^5 + k^6$

Write under each term the sum of the product of its coeff^t. and the no. of terms from the left and the product of the coeff^t. of the preceding term and its no. of terms from above.

i. $B_{2n} = I_{2n} + (-1)^n (F_{2n} - 1)$

where I_{2n} is ~~the nearest~~ integer to B_{2n} and F_{2n} is the sum of the reciprocals of prime no.s next to the factors of $2n$ including unity and the number itself.

ii. The numerator of B_n is divisible by the greatest odd multiple of n prime to $(2^n - 1)$.

$I_0 = I_2 = I_4 = I_6 = I_8 = I_{10} = I_{12} = 0.$

$I_{14} = 1$, $I_{16} = 7$, $I_{18} = 55$, $I_{20} = 529$.

$I_{22} = 6192$, $I_{24} = 86580$, $I_{26} = 1425517$

Cor 6. $\psi_n(x-1)$ is the integral part of

$$\frac{x^{n+1}}{1-x}\left\{\frac{\lfloor n}{\left(\log_e\frac{1}{1-x}\right)^{n+1}} - \frac{B_{n+1}}{n+1}\sin\frac{\pi n}{2}\right\}$$

Sol. $e^{-x}+e^{-2x}+e^{-3x}+\&c = \frac{1}{e^x-1} = \frac{1}{x} - \&c$

Differentiating n times we have

$1^n e^{-x} + 2^n e^{-2x} + 3^n e^{-3x} + \&c = \frac{\lfloor n}{x^{n+1}} \pm \&c$

Writing $\log_e \frac{1}{1-x}$ for x we have

$$\frac{1^n}{1-x}+\frac{2^n}{(1-x)^2}+\frac{3^n}{(1-x)^3}+\&c = \frac{\lfloor n}{\left(\log_e\frac{1}{1-x}\right)^{n+1}} \pm \&c$$

Applying VI 2 Cor 1. we at once get the result.

Ex. 1. Shew that f(x) is the term independent of n in

$$\frac{\phi(x)+\frac{1}{n}\phi'(x)+\frac{1}{n^2}\phi''(x)+\frac{1}{n^3}\phi'''(x)+\&c}{e^{nh}+k}$$

2. If $n \not> r$, $F_r(n-1)$ is the coeft. of $\frac{x^{n-1}}{\lfloor n-1}$ in $e^{x(r-n)}(e^x-1)^n$

3. $$\frac{\cos x + k}{1+2k\cos x+k^2} = \frac{\psi_0(k)}{k+1} - \frac{x^2}{\lfloor 2}\frac{\psi_2(k)}{(k+1)^3} + \frac{x^4}{\lfloor 4}\frac{\psi_4(k)}{(k+1)^5} - \&c$$

4. $$\frac{\sin x}{1+2k\cos x+k^2} = \frac{x}{\lfloor 1}\frac{\psi_1(k)}{(k+1)^2} - \frac{x^3}{\lfloor 3}\frac{\psi_3(k)}{(k+1)^4} + \frac{x^5}{\lfloor 5}\frac{\psi_5(k)}{(k+1)^6} - \&c$$

5. If n is even shew that $\psi_n(k)$ is divisible by $(1-k)$

6. If $1^n(S_2-1) - 2^n(S_3-1) + 3^n(S_4-1) - \&c = \cos\pi n -$
$\sin\frac{\pi n}{2}\frac{B_{n+1}}{n+1}(2^{n+1}-1) + A_1 S_2 - A_2 S_3 + A_3 S_4 - \&c$ where
$S_n = \frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\&c$, then

i. $A_r + r A_{r-1} + \frac{r(r-1)}{\lfloor 2}A_{r-2} + \frac{r(r-1)(r-2)}{\lfloor 3}A_{r-3} + \&c + A_0 = r^n$

ii. $A_r = r^n - r(r-1)^n + \frac{r(r-1)}{\lfloor 2}(r-2)^n - \&c$

iii. $\frac{A_r}{\lfloor n}$ is the coeft. of x^n in $(e^x-1)^r$

$$\frac{3392780147 + 6960\left\{\frac{1^{27}x}{1-x} + \frac{2^{27}x^2}{1-x^2} + \frac{3^{27}x^3}{1-x^3} + \&c\right\}}{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \&c\right)}$$

*

$$= 489693897\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \&c\right)\right\}^6$$

$$+ 2507636250\left\{1 + 240\left(\frac{1^3x}{1-x} + \&c\right)\right\}^3\left\{1 - 504\left(\frac{1^5x}{1-x} + \&c\right)\right\}^2$$

$$+ 395450000\left\{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \&c\right)\right\}^4.$$

$$B_{38} = \frac{2929,993913,841559}{6}.$$

$$\frac{1723168255201 - 171864\left\{\frac{1^{29}x}{1-x} + \frac{2^{29}x^2}{1-x^2} + \frac{3^{29}x^3}{1-x^3} + \&c\right\}}{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \&c\right)}.$$

iv. $\psi_n(K-1) = A_n - A_{n-1}K + A_{n-2}K^2 - \&c$ to n terms.

7. Shew that

i. $\frac{1^5}{2} + \frac{2^5}{2^2} + \frac{3^5}{2^3} + \frac{4^5}{2^4} + \&c = 1082.$

ii. $\frac{1^5}{3} + \frac{2^5}{3^2} + \frac{3^5}{3^3} + \frac{4^5}{3^4} + \&c = 68\frac{1}{4}.$

5. $\frac{x}{e^x - 1} = 1 - \frac{x}{2} + B_2\frac{x^2}{\underline{|2}} - B_4\frac{x^4}{\underline{|4}} + B_6\frac{x^6}{\underline{|6}} - \&c$

Sol. Write e^x for $\phi(x)$ in VI

Cor 1. $\frac{x}{e^x+1} = \frac{x}{2} - B_2\frac{x^2}{\underline{|2}}(2^2-1) + B_4\frac{x^4}{\underline{|4}}(2^4-1) - \frac{B_6}{\underline{|6}}x^6(2^6-1) + \&c$

Sol. $\frac{x}{e^x+1} = \frac{x}{e^x-1} - \frac{2x}{e^{2x}-1}$

Cor 2. $\log_e\frac{x}{e^x-1} = -\frac{x}{2} - B_2\frac{x^2}{2\underline{|2}} + B_4\frac{x^4}{4\underline{|4}} - B_6\frac{x^6}{6\underline{|6}} + \&c$

Sol. $\log_e(e^x-1) = \int\frac{e^x}{e^x-1}\,dx$

Cor 3. $\log_e\frac{2}{e^x+1} = -\frac{x}{2} - B_2\frac{x^2}{2\underline{|2}}(2^2-1) + B_4\frac{x^4}{4\underline{|4}}(2^4-1) - \&c$

Sol. $\log_e(e^x+1) = \log_e(e^{2x}-1) - \log_e(e^x-1)$

Ex. If P, Q, R, S &c are so small that $\frac{1}{720}$ of the sum of their cubes may be neglected shew that

1. If $e^P + e^Q + e^R = 2 + e^{P+Q+R}$ then

$$\frac{1}{P} + \frac{1}{Q} + \frac{1}{R} + \frac{1}{2} + \frac{P+Q+R}{12} = 0$$

2. If $e^{P+Q+R+S} = \frac{e^P + e^Q + e^R + e^S - 2}{e^{-P} + e^{-Q} + e^{-R} + e^{-S} - 2}$ then

$$\frac{1}{P} + \frac{1}{Q} + \frac{1}{R} + \frac{1}{S} \quad \frac{P+Q+R+S}{12}$$

$$B_{22} = \frac{11(57183 + 20500)}{138}$$

$$B_{24} = \frac{236364091}{2730} = \frac{19.1617^2 + 10.4200^2 + 34.550^2}{2730}$$

$$B_{26} = \frac{8553103}{6} = \frac{13(392931 + 265000)}{6}$$

$$236364091 + 131040\left(\frac{1^{23}x}{1-x} + \frac{2^{23}x^2}{1-x^2} + \&c\right)$$

$$= 49679091\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \&c\right)\right\}^6$$

$$+ 176400000\left\{1 + 240\left(\frac{1^3x}{1-x} + \&c\right)\right\}^3\left\{1 - 504\left(\frac{1^5x}{1-x} + \&c\right)\right\}^2$$

$$+ 10285000\left\{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \&c\right)\right\}^4.$$

$$B_{28} = \frac{23749461029}{870} = \frac{7}{870}\left(19.23.11^2.21^3 + 2.525^2.4549 + 55.10^4.719\right)$$

$$B_{30} = \frac{8615841276005}{14322}.$$

$$B_{32} = \frac{7709321041217}{510}$$

$$B_{34} = \frac{2577687858367}{6}.$$

$$B_{36} = \frac{26315271553053477373}{1919190}$$

3. If $2e^{P+Q+R+S+T} = \dfrac{(e^P+e^Q+e^R+e^S+e^T-2)^2-(e^{2P}+e^{2Q}+e^{2R}+e^{2S}+e^{2T}-2)}{e^{-P}+e^{-Q}+e^{-R}+e^{-S}+e^{-T}-2}$, then

$$\frac{1}{P}+\frac{1}{Q}+\frac{1}{R}+\frac{1}{S}+\frac{1}{T}+\frac{P+Q+R+S+T}{12}=\frac{1}{2}$$

6. $x\cot x = 1 - B_2\dfrac{(2x)^2}{\lfloor 2} - B_4\dfrac{(2x)^4}{\lfloor 4} - B_6\dfrac{(2x)^6}{\lfloor 6} - B_8\dfrac{(2x)^8}{\lfloor 8}$ &c

Sol. Write xi for x in VI 5.

N.B. From the nature of the coeff^ts we see that $B_0 = -1$

Cor. $(2n+1)B_{2n} = 2B_2B_{2n-2}\dfrac{2n(2n-1)}{\lfloor 2} + 2B_4B_{2n-4}\dfrac{2n(2n-1)(2n-2)(2n-3)}{\lfloor 4}$ + &c the last term being $2B_{n-1}B_{n+1}\dfrac{\lfloor 2n}{\lfloor n-1\,\lfloor n+1}$ or $(B_n)^2\dfrac{\lfloor 2n}{(\lfloor n)^2}$ according as n is odd or even

Sol. $\cot^2 x = -\left(1+\dfrac{d\cot x}{dx}\right)$ Equate the coeff^ts of x^{2n}

$B_0=-1$, $B_2=\frac{1}{6}$, $B_4=\frac{1}{30}$, $B_6=\frac{1}{42}$, $B_8=\frac{1}{30}$, $B_{10}=\frac{5}{66}$, $B_{12}=\frac{691}{2730}$

$B_{14}=\frac{7}{6}$, $B_{16}=\frac{3617}{510}$, $B_{18}=\frac{43867}{798}$, $B_{20}=\frac{174611}{330}$, $B_{22}=\frac{854513}{138}$

1. $x\,\mathrm{cosec}\,x = 1 + B_2\dfrac{x^2(2^2-2)}{\lfloor 2} + B_4\dfrac{x^4(2^4-2)}{\lfloor 4}$ + &c

Sol. $\mathrm{cosec}\,x = \cot\frac{x}{2} - \cot x$

2. $x\tan x = B_2\dfrac{2^2-1}{\lfloor 2}(2x)^2 + B_4\dfrac{2^4-1}{\lfloor 4}(2x)^4$ + &c

Sol. $\tan x = \cot x - 2\cot 2x$

3. $\log_e(x\,\mathrm{cosec}\,x) = B_2\dfrac{(2x)^2}{2\lfloor 2} + B_4\dfrac{(2x)^4}{4\lfloor 4} + B_6\dfrac{(2x)^6}{6\lfloor 6}$ + &c

Sol. $\log_e \sin x = \int \cot x\,dx$.

4. $\log_e \sec x = B_2\dfrac{2^2-1}{2\lfloor 2}(2x)^2 + B_4\dfrac{2^4-1}{4\lfloor 4}(2x)^4$ [illegible]

Sol. $\log_e \sec x = \int \tan x\,dx$

If $\alpha, \beta, \gamma, \delta$ &c are the roots of the eq^n: $f(x)=0$

then $f(x) = f(0)\left(1-\frac{x}{\alpha}\right)\left(1-\frac{x}{\beta}\right)\left(1-\frac{x}{\gamma}\right)$ &c

$$\pi x\left\{1+\left(\frac{x}{n+1}\right)^2\right\}\left\{1+\left(\frac{x}{n+2}\right)^2\right\}\left\{1+\left(\frac{x}{n+3}\right)^2\right\} \&c$$

$$= \left(\frac{\lfloor n}{x^n}\right)^2 \sqrt{\frac{\cosh 2\pi x - \cos 2\pi x}{2}}^{\,*}\, e^{-\frac{S_2}{x^2}+\frac{S_4}{2x^4}-\frac{S_6}{3x^6}+\&c}$$

where $S_p = 1^p + 2^p + 3^p + \cdots n^p$.

$$\pi(x^2+n^2)^{n+\frac{1}{2}}\left\{1+\left(\frac{x}{n+1}\right)^2\right\}\left\{1+\left(\frac{x}{n+2}\right)^2\right\}\left\{1+\left(\frac{x}{n+3}\right)^2\right\} \&c$$

$$= (\lfloor n)^2 \sqrt{\frac{\cosh 2\pi x - \cos 2\pi x}{2}}^{\,*}\, e^{2n - 2x\tan^{-1}\frac{n}{x} - \frac{B_2 S_2}{x} - \frac{B_4 S_4}{2x^3}}$$

$$- \frac{B_6 S_6}{3x^5} - \&c \quad \text{where } S_p = \frac{n}{x} - \frac{p(p+1)}{\lfloor 3}\left(\frac{n}{x}\right)^3 + \frac{p(p+1)(p+2)(p+3)}{\lfloor 5} \times \left(\frac{n}{x}\right)^5 - \&c$$

$$2\pi(n^2+x^2)^{n-\frac{1}{2}}\left\{1+\left(\frac{x}{n}\right)^2\right\}\left\{1+\left(\frac{x}{n+1}\right)^2\right\}\left\{1+\left(\frac{x}{n+2}\right)^2\right\}$$

$$= (\lfloor n-1)^2 e^{2n+2x\theta - \frac{2B_2\cos\theta}{1.2r} + \frac{2B_4\cos 3\theta}{3.4r^3} - \&c} \times \left(1 - e^{-2\pi x}E.\right)$$

where $r^2 = n^2+x^2$, $\tan\theta = \frac{x}{n}$ & E the error is less than 1 & $= \cos 2\pi n$ ~~when x is~~

7. $\operatorname{Sec} x = E_1 + \frac{x^2}{\lfloor 2} E_3 + \frac{x^4}{\lfloor 4} E_5 + \frac{x^6}{\lfloor 6} E_7 + \&c$

Cor. $\frac{B_{2n}}{2n} 2^{2n}(2^{2n}-1) = 2E_1E_{2n-1} + 2E_3E_{2n-3}\frac{(2n-2)(2n-3)}{\lfloor 2} + \&c$

the last term being $2E_{n-1}E_{n+1}\frac{\lfloor 2n-2}{\lfloor n-2\,\lfloor n}$ or $(E_n)^2\frac{\lfloor 2n-2}{(\lfloor n-1)^2}$

according as n is even or odd

Sol. $\frac{d\tan x}{dx} = \operatorname{Sec}^2 x$. Equate the coeff^ts of x^{2n-2}

$E_1 = 1$, $E_3 = 1$, $E_5 = 5$, $E_7 = 61$, $E_9 = 1385$, $E_{11} = 50521$,

$E_{13} = 2702765$, $E_{15} = 199360981$, &c

8. i. $\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)$ &c

Sol. The roots of the equation $\frac{\sin x}{x} = 0$ are $\pm\pi, \pm 2\pi$, &c

and $\frac{\sin x}{x} = 1$ when $x = 0$

ii. In a similar manner

$\cos x = \left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{3^2\pi^2}\right)\left(1 - \frac{4x^2}{5^2\pi^2}\right)\left(1 - \frac{4x^2}{7^2\pi^2}\right)$ &c

Cor 1. $\frac{e^x - e^{-x}}{2x} = \left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{2^2\pi^2}\right)\left(1 + \frac{x^2}{3^2\pi^2}\right)$ &c

2. $\frac{e^x + e^{-x}}{2} = \left(1 + \frac{4x^2}{\pi^2}\right)\left(1 + \frac{4x^2}{3^2\pi^2}\right)\left(1 + \frac{4x^2}{5^2\pi^2}\right)$ &c

3. $\cos x + \sin x = \left(1 + \frac{4x}{\pi}\right)\left(1 - \frac{4x}{3\pi}\right)\left(1 + \frac{4x}{5\pi}\right)$ &c

Ex 1. $\frac{\sin(x+a)}{\sin a} = \left(1 + \frac{x}{a}\right)\left(1 - \frac{x}{\pi - a}\right)\left(1 + \frac{x}{\pi + a}\right)\left(1 - \frac{x}{2\pi - a}\right) \times$

$\left(1 + \frac{x}{2\pi + a}\right)\left(1 - \frac{x}{3\pi - a}\right)\left(1 + \frac{x}{3\pi + a}\right)$ &c

2. $\frac{\cos(x+a)}{\cos a} = \left(1 + \frac{x}{\frac{\pi}{2} + a}\right)\left(1 - \frac{x}{\frac{\pi}{2} - a}\right)\left(1 + \frac{x}{\frac{3\pi}{2} + a}\right) \times$

$\left(1 - \frac{x}{\frac{3\pi}{2} - a}\right)$ &c

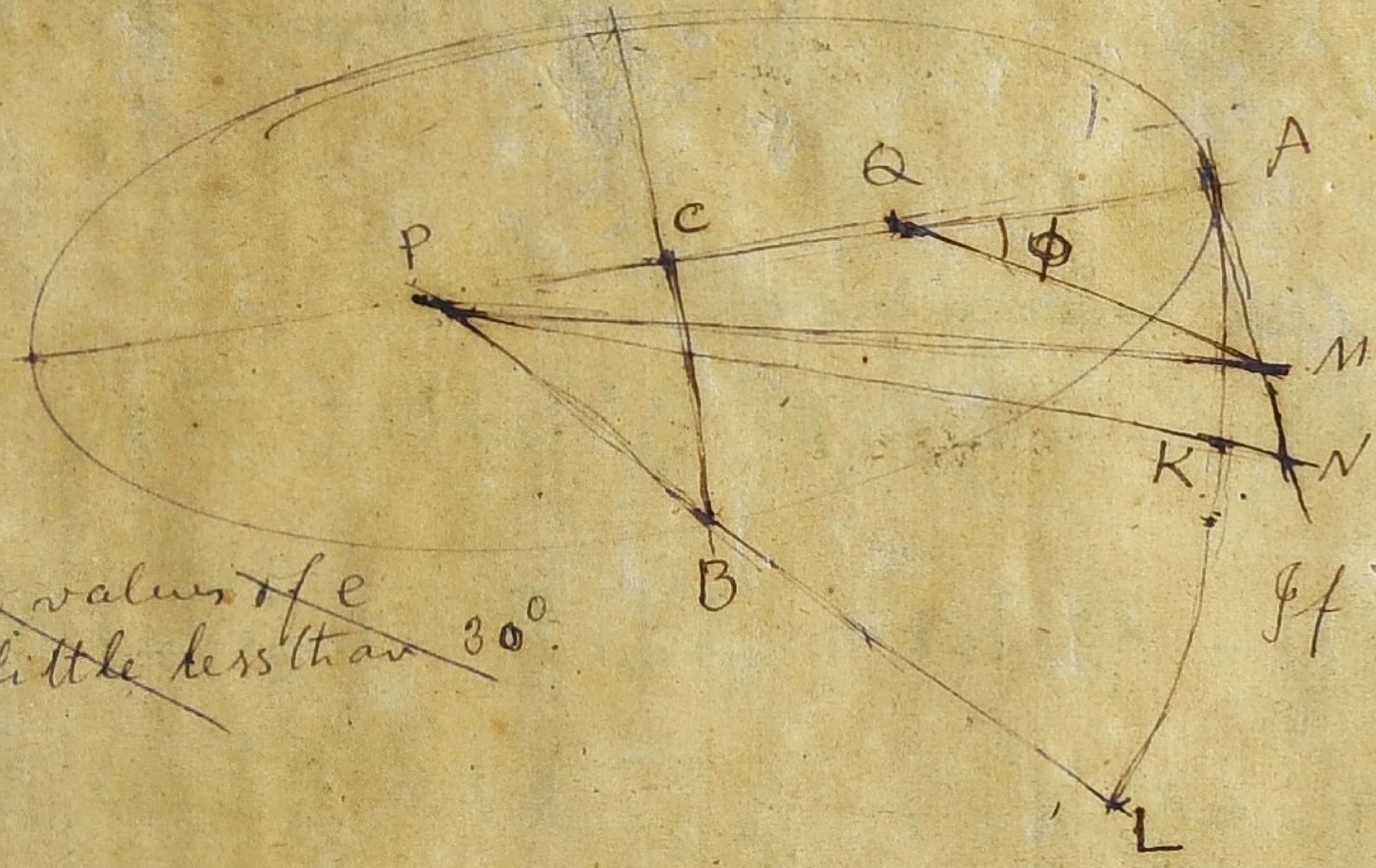
3. $1 + \frac{\sin x}{\sin a} = \left(1 + \frac{x}{a}\right)\left(1 + \frac{x}{\pi - a}\right)\left(1 - \frac{x}{\pi + a}\right)\left(1 - \frac{x}{2\pi - a}\right)$ &c

$$\int_0 \frac{\sin x}{x}\,dx = \frac{\pi}{2} - r\cos(x-\theta).$$

$$\int_0 \frac{1-\cos x}{x}\,dx = C + \log x - r\sin(x-\theta).$$

where $r^2 = \frac{\lfloor 1}{x^2} - \frac{\lfloor 3}{2x^4} + \frac{\lfloor 5}{3x^6} - \frac{\lfloor 7}{4x^8} + \&c$

$$\left.\begin{aligned} r\cos\theta &= \frac{1}{x} - \frac{\lfloor 2}{x^3} + \frac{\lfloor 4}{x^5} - \&c \\ r\sin\theta &= \frac{\lfloor 1}{x^2} - \frac{\lfloor 3}{x^4} + \frac{\lfloor 5}{x^6} - \&c \end{aligned}\right\}$$



$$\left\{\begin{aligned} &\angle MPN = \tfrac{1}{2}\angle APN. \\ &CP = CB = CQ \end{aligned}\right.$$

$$\frac{AL}{AK} = \frac{AB}{AN}$$

If $e = 0{\cdot}21$ $\phi = 30^0$

~~For other values of e ϕ is a little less than 30^0.~~

$$\pi(a+b)\left\{1 + \frac{\sin^2\theta}{2+\cos^2\frac{\theta}{2}}\right\} \text{ where } \sin\theta = \frac{a-b}{a+b}\sin\phi.$$

when $e = 1$ $3\phi - \pi = 0^0 - 14' - 46''$

ϕ rapidly diminishes to 60^0.

$$\pi(a+b)\left\{1 + 4\sin^2\frac{\theta}{2}\right\} \text{ where } \sin\theta = \frac{a-b}{a+b}\sin\phi$$

when $e = 1$ $3\phi - \frac{\pi}{2} = 0^0 - 54' - 19''$

8. $1+\frac{\sin x}{\cos a} = \left(1+\frac{x}{\frac{\pi}{2}-a}\right)\left(1+\frac{x}{\frac{\pi}{2}+a}\right)\left(1-\frac{x}{\frac{3\pi}{2}-a}\right)\left(1+\frac{x}{\frac{3\pi}{2}+a}\right)$ &c

N.B. If we write the above results as

$\phi(x) = (1+a_1x)(1+a_2x)(1+a_3x)$ &c, then it is possible to

find $(1+a_1^n x^n)(1+a_2^n x^n)(1+a_3^n x^n)$ &c

9. $\cot x = \frac{1}{x} - \frac{1}{\pi-x} + \frac{1}{\pi+x} - \frac{1}{2\pi-x} + \frac{1}{2\pi+x} -$ &c

Sol. Equate the coeffts. of x in VI 8 Ex 1.

Cor 1. $\tan x = \frac{1}{\frac{\pi}{2}-x} - \frac{1}{\frac{\pi}{2}+x} + \frac{1}{\frac{3\pi}{2}-x} - \frac{1}{\frac{3\pi}{2}+x} +$ &c

2. $\operatorname{cosec} x = \frac{1}{x} + \frac{1}{\pi-x} - \frac{1}{\pi+x} - \frac{1}{2\pi-x} +$ &c

3. $\sec x = \frac{1}{\frac{\pi}{2}-x} + \frac{1}{\frac{\pi}{2}+x} - \frac{1}{\frac{3\pi}{2}-x} - \frac{1}{\frac{3\pi}{2}+x} +$ &c

10. $\tan^{-1}\frac{x}{a} - \tan^{-1}\frac{x}{\pi-a} + \tan^{-1}\frac{x}{\pi+a} -$ &c $= \tan^{-1}\left\{\frac{e^x-e^{-x}}{e^x+e^{-x}}\cot a\right\}$

i.e. L.H.S $= \frac{1}{2i}\log_e\left\{\frac{1+\frac{xi}{a}}{1-\frac{xi}{a}}\cdot\frac{1-\frac{xi}{\pi-a}}{1+\frac{xi}{\pi-a}}\ \text{\&c}\right\}$ Apply VI 8 Ex 1.

Cor 1. $\tan^{-1}\frac{x}{\frac{\pi}{2}-a} - \tan^{-1}\frac{x}{\frac{\pi}{2}+a} + \tan^{-1}\frac{x}{\frac{3\pi}{2}-a} -$ &c $=$

$\tan^{-1}\left\{\frac{e^x-e^{-x}}{e^x+e^{-x}}\tan a\right\}$

Cor 2. $\tan^{-1}\frac{x}{1} - \tan^{-1}\frac{x}{3} + \tan^{-1}\frac{x}{5} -$ &c $= \tan^{-1}\left(\frac{e^{\frac{\pi x}{2}}-1}{e^{\frac{\pi x}{2}}+1}\right)$

Cor 3. $\tan^{-1}\frac{x}{a} + \tan^{-1}\frac{x}{\pi-a} - \tan^{-1}\frac{x}{\pi+a} -$ &c $= \tan^{-1}\left\{\frac{e^x-e^{-x}}{2}\operatorname{cosec} a\right\}$

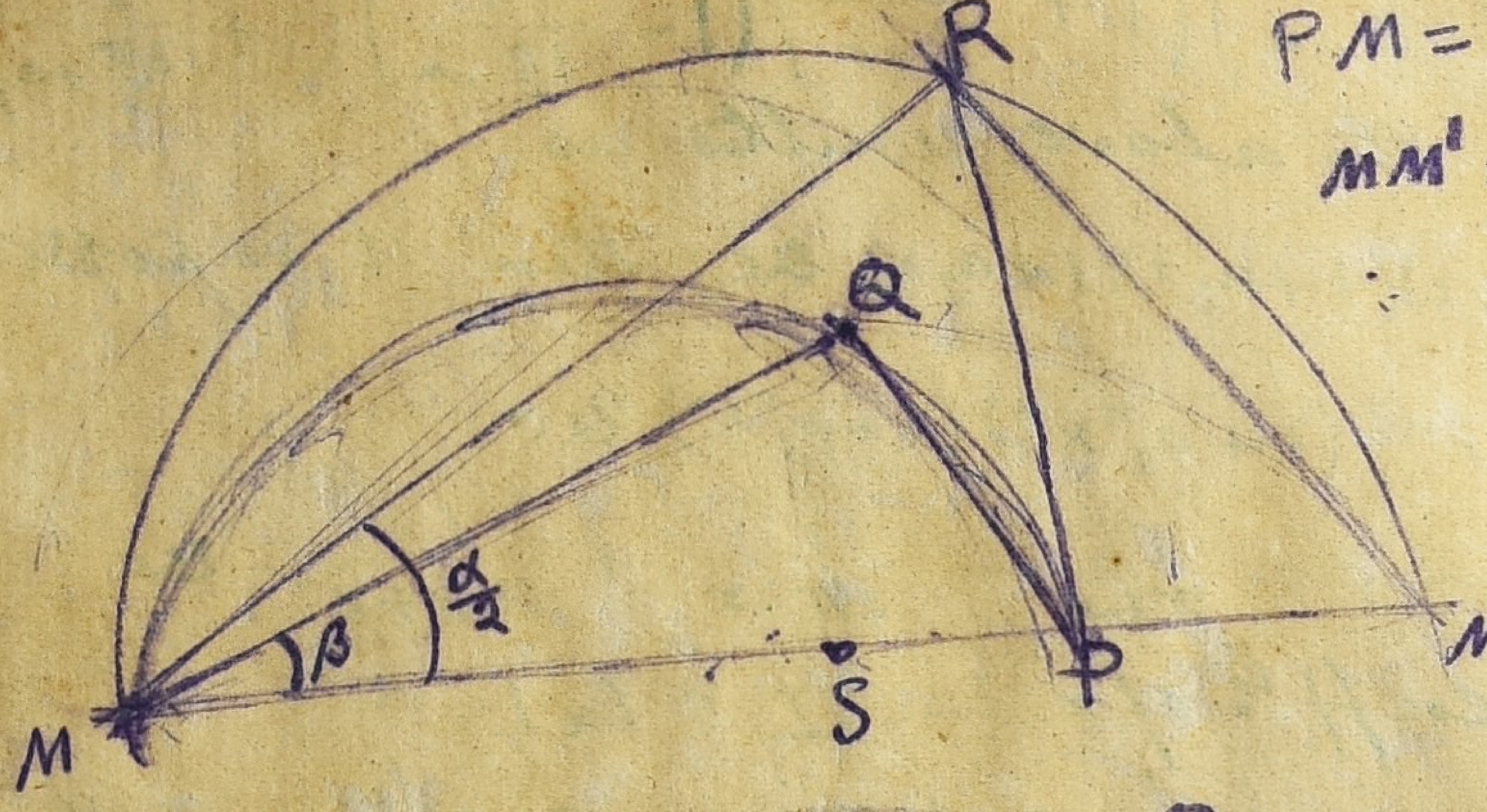
Cor 4. $\tan^{-1}\frac{x}{\frac{\pi}{2}-a} + \tan^{-1}\frac{x}{\frac{\pi}{2}+a} - \tan^{-1}\frac{x}{\frac{3\pi}{2}-a} -$ &c $= \tan^{-1}\left\{\frac{e^x-e^{-x}}{2}\sec a\right\}$

Cor 5. $\tan^{-1}\frac{x}{1} + \tan^{-1}\frac{x}{3} - \tan^{-1}\frac{x}{5} - \tan^{-1}\frac{x}{7}$ &c

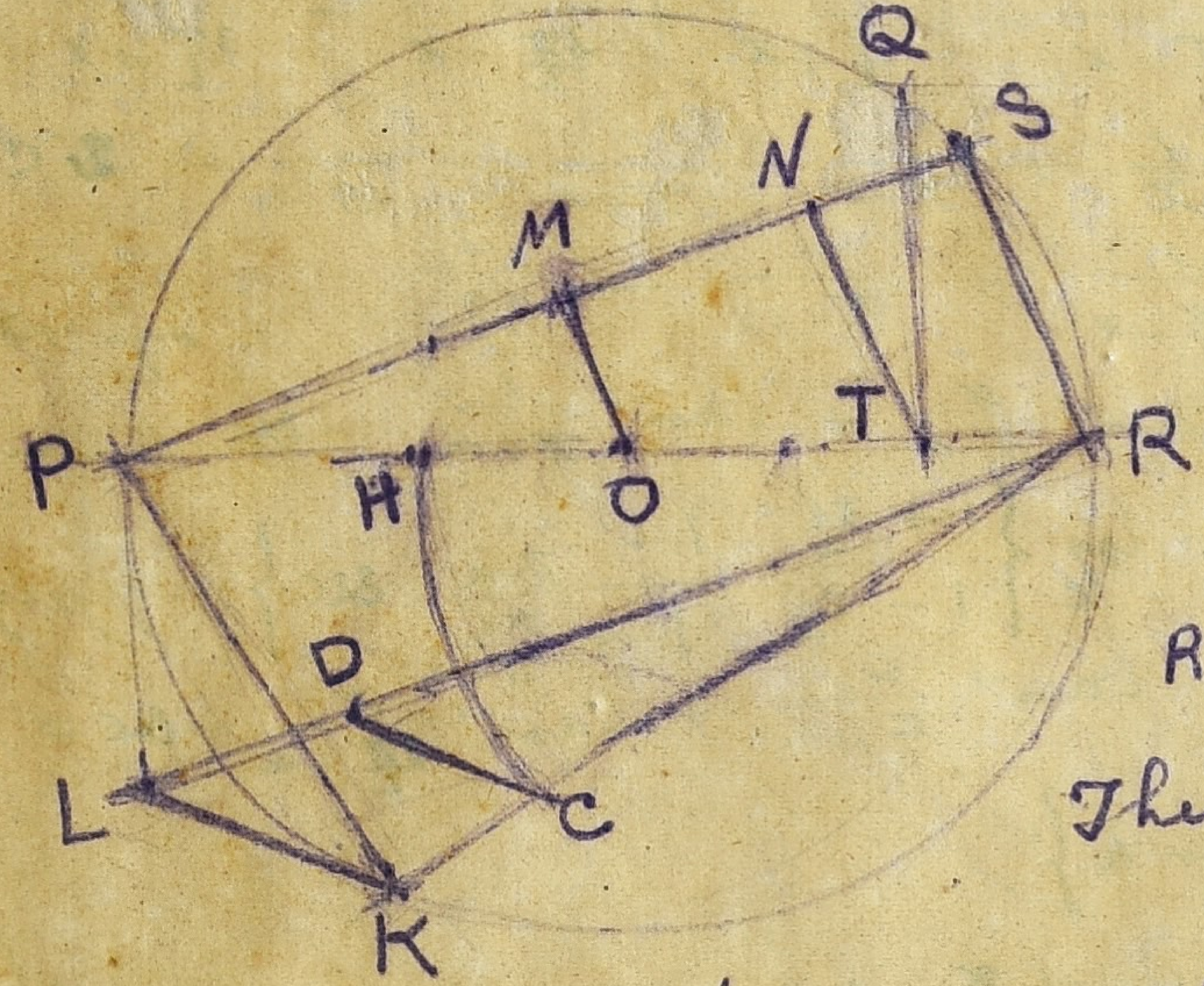
$\tan^{-1}\left(\frac{e^{\frac{\pi x}{2}}-1}{\sqrt{2e^{\frac{\pi}{2}x}}}\right)$

Ex. 1. $\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}$ &c

2. $\sqrt{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{10}{9}\cdot\frac{10}{11}$ &c



$PM = 1$; $PQ = PM'$.
$MM' = m$; $MS = M'S$.
$\therefore\ 2PS = m \cos\alpha$.



$OH = \frac{1}{2} OP$ }
$RT = \frac{1}{3} OR$ }
$RS = TQ$
OM, TN, RS are ∥.
$PK = PM$ & $PL = MN$.
$RC = RH$ & CD is ∥^{l} to LK
Then $RD^2 = \odot$

~~From RD cut off $\frac{1}{23275366}$th of it.~~

$$\pi = \frac{355}{113}\left(1 - \frac{.0003}{3533}\right).$$

Perimeter of ~~an~~ ellipse $= \pi(a+b)\left\{1 + \frac{3t}{10 + \sqrt{4-3t}}\right\}$ very nearly

where $t = \left(\frac{a-b}{a+b}\right)^2$.

CHAPTER VII

1\. i. $\frac{1}{1^2-x^2}+\frac{1}{2^2-x^2}+\frac{1}{3^2-x^2}+\frac{1}{4^2-x^2}+\&c=\frac{1}{2x^2}-\frac{\pi}{2x}\cot\pi x.$

ii. $\frac{1}{1^2-x^2}+\frac{1}{3^2-x^2}+\frac{1}{5^2-x^2}+\frac{1}{7^2-x^2}+\&c=\frac{\pi}{4x}\tan\frac{\pi x}{2}.$

iii. $\frac{1}{1^2-x^2}-\frac{1}{2^2-x^2}+\frac{1}{3^2-x^2}-\frac{1}{4^2-x^2}+\&c=\frac{\pi}{2x}\operatorname{cosec}\pi x-\frac{1}{2x^2}$

iv. $\frac{1}{1^2-x^2}-\frac{3}{3^2-x^2}+\frac{5}{5^2-x^2}-\&c=\frac{\pi}{4}\sec\frac{\pi x}{2}.$

Ex. 1. $\frac{1}{1^2+x^2}+\frac{1}{2^2+x^2}+\frac{1}{3^2+x^2}+\&c=\frac{\pi}{2x}\,\frac{e^{\pi x}+e^{-\pi x}}{e^{\pi x}-e^{-\pi x}}-\frac{1}{2x^2}$

2\. $\frac{1}{1^2+x^2}+\frac{1}{3^2+x^2}+\frac{1}{5^2+x^2}+\&c=\frac{\pi}{4x}\,\frac{e^{\pi x}-1}{e^{\pi x}+1}.$

3\. $\frac{1}{1^2+x^2}-\frac{1}{2^2+x^2}+\frac{1}{3^2+x^2}-\&c=\frac{1}{2x^2}-\frac{\pi}{x(e^{\pi x}-e^{-\pi x})}$

4\. $\frac{1}{1^2+x^2}-\frac{3}{3^2+x^2}+\frac{5}{5^2+x^2}-\frac{7}{7^2+x^2}+\&c=\frac{\frac{\pi}{2}}{e^{\frac{\pi x}{2}}+e^{-\frac{\pi x}{2}}}$

2\. i. $\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\&c=\frac{(2\pi)^{2n}}{2\,|\underline{2n}}B_{2n}$

ii. $\frac{1}{1^{2n}}+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\&c=\frac{(2^{2n}-1)\pi^{2n}}{2\,|\underline{2n}}B_{2n}$

iii. $\frac{1}{1^{2n}}-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\&c=\frac{(2^{2n}-2)\pi^{2n}}{2\,|\underline{2n}}B_{2n}$

iv. $\frac{1}{1^{2n-1}}-\frac{1}{3^{2n-1}}+\frac{1}{5^{2n-1}}-\&c=\frac{\pi^{2n-1}}{2^{2n}\,|\underline{2n-2}}E_{2n-1}$

N.B. From Chap. VI we know the values of B_n for even integers and E_n for odd integers, we know that B_n and E_n are infinite when $n=\infty$, that the values of B_n are all fractions though $2(2^n-1)B_n$ is an integer n being even and those of E_n are all integers n being odd, in both cases n being positive. Now let us try to interpret some meaning for B_n and E_n when n is positive, integral or fractional

If $l = (a-b)\cos\phi = (a+b)\tan\theta$

and $\frac{\pi l}{\theta}$ be the perimeter of the ellipse.

then $\phi = \frac{2\sqrt{ab}}{a+b}\left\{30^\circ + 6^\circ\,18'\,49'' \frac{(\sqrt{a}-\sqrt{b})^2}{a+b} - 1^\circ\,10'\,55'' \left(\frac{a-b}{a+b}\right)^2\right\}$

(ϕ in the figure).

~~$3\phi - 90^\circ = \frac{4ab}{(a+b)^2}\cdot\left(\frac{a-b}{a+b}\right)^2\left\{72^\circ\,42'\,3''\cdot\frac{ab}{(a+b)^2} - 2^\circ\,12'\,14''\right\}$~~

$\phi = 30^\circ + h(1-h)\left\{5^\circ\,19\tfrac{1}{2}' - 6^\circ\,3\tfrac{1}{2}'\,h\right\}$

where $h = \left(\frac{a-b}{a+b}\right)^2$

Assume VII 2 to be true for all positive values of n.
To find B_n and E_n for negative values of n see Chap - ter IX.

3. $\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\&c = S_n = \frac{(2\pi)^n}{2\,|\underline{n}}B_n$

From this we can find B_n when $n>1$.

$B_1=\infty$, $B_3=\frac{3}{2\pi^3}S_3$, &c

$B_{1\frac{1}{2}}=\frac{3}{4\pi\sqrt{2}}S_{1\frac{1}{2}}$ &c.

N.B. To find $|\underline{n}$ for fractional values of n see Chap.

4. $\frac{1}{1^n}-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\&c = \frac{(2^{n-1}-1)\pi^n}{|\underline{n}}B_n$

From this we can find B_n if n is not negative.

$B_0=-1$, $B_{\frac{1}{2}}=-\left(1+\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\&c\right)$

5. $\frac{1}{1^n}-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\&c = \frac{\pi^n}{2^{n+1}|\underline{n-1}}E_n$

From this we can find E_n if n is not negative

$E_0=\infty$, $E_{\frac{1}{2}}=2\sqrt{2}\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{7}}+\&c\right)$

$E_2=\frac{8}{\pi^2}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\&c\right)$.

6. $\frac{1}{(a+b)^n}+\frac{1}{(a+2b)^n}+\frac{1}{(a+3b)^n}+\&c = \frac{1}{b(n-1)a^{n-1}}-\frac{1}{2a^n}$

$+B_2\frac{n}{|\underline{2}}\cdot\frac{b}{a^{n+1}}-B_4\frac{n(n+1)(n+2)}{|\underline{4}}\cdot\frac{b^3}{a^{n+3}}+\&c$

From this we can sum up the reciprocals of powers of all numbers in A P approximately

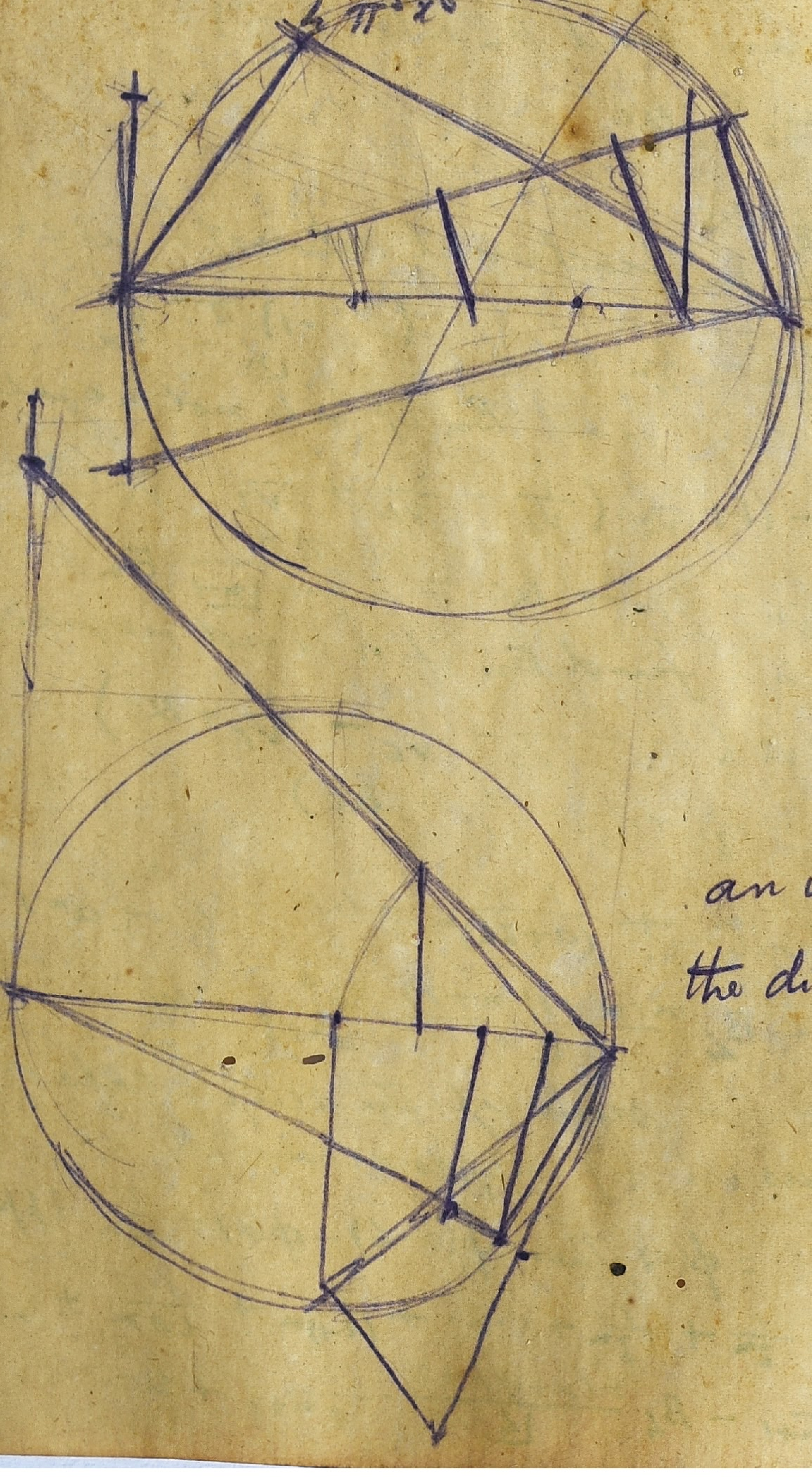
Sol. Let L. H. S $=\phi(a)$, then $\phi(a-b)-\phi(a)=\frac{1}{a^n}$ Apply VI

Cor. $S_n = \frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\dots+\frac{1}{(R-1)^n}+\frac{1}{2R^n}+\frac{1}{(n-1)R^{n-1}}$

$+B_2\frac{n}{|\underline{2}}\cdot\frac{1}{R^{n+1}}-B_4\frac{n(n+1)(n+2)}{|\underline{4}}\cdot\frac{1}{R^{n+3}}+\&c$

$$\left(1+\frac{x^6}{1^6}\right)\left(1+\frac{x^6}{2^6}\right)\left(1+\frac{x^6}{3^6}\right) \&c$$

$$= \frac{\sinh 2\pi x - 2\sinh \pi x \cos \pi x\sqrt{3}}{4\pi^3 x^3}.$$



an inch greater if
the diameter be 5000 miles
long

$S_2 = 1.6449340668$ $\frac{1}{B_1} = 0$, $\frac{1}{B_2} = 6$

$S_3 = 1.2020569031$ $\frac{1}{B_3} = 17.19624$

$S_4 = 1.0823232337$ $\frac{1}{B_4} = 30$, $\frac{1}{B_5} = 39.34953$

$S_5 = 1.0369277551$

$S_6 = 1.0173430620$ $\frac{1}{B_6} = 42$ $\frac{1}{B_7} = 38.03538$

$S_7 = 1.0083492774$

$S_8 = 1.0040773562$ $\frac{1}{B_8} = 30$ $\frac{1}{B_9} = 20.98719$

$S_9 = 1.0020083928$ $\frac{1}{B_{10}} = 13.2$

$S_{10} = 1.0009945781$

Ex. 1. Shew that when $n = 0$ $\left(\frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \&c\right) - \frac{1}{n}$ is a finite quantity the value of which is $\cdot 57$ nearly and $n S_{n+1} = 1$

Sol. In VII 6 Cor write $n+1$ for n and 1 for x; then we have $S_{n+1} - \frac{1}{n} = \frac{1}{2} + B_2 \frac{n+1}{\lfloor 2} - \&c.$

∴ when $n = 0$ $S_{n+1} - \frac{1}{n} = \frac{1}{2} + \frac{1}{12} - \frac{1}{120} + \&c$

$= \cdot 577$ nearly.

2. Shew that $\pi n B_{n+1} = 1$ when $n = 0$.

Sol. We have $S_{n+1} = \frac{(2\pi)^{n+1}}{2\lfloor n+1} B_{n+1}$. Multiplying both sides by n and then writing 0 for n, we have $1 = \frac{(2\pi)^{n+1}}{2\lfloor n+1} n B_{n+1}$ when $n = 0$.

i.e. when $n = 0$ $\pi n B_{n+1} = 1$

3. Shew that $\frac{1 + E_2}{1 - E_2} = \frac{\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{9^2} + \frac{1}{13^2} + \&c}{\frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{15^2} + \&c}$

Sol. $E_2 = \frac{8}{\pi^2}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c\right)$

$= \frac{\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c}{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \&c}$ Apply Componendo & dividendo

$$(1-x^2)(1-x^3)(1-x^5)(1-x^7)(1-x^{11})(1-x^{13})(1-x^{17})\ \&c$$

$$= 1 + \frac{x^2}{1-x} + \frac{x^5}{(1-x)(1-x^2)} + \frac{x^{10}}{(1-x)(1-x^2)(1-x^3)}$$

$$+ \frac{x^{17}}{(1-x)(1-x^2)(1-x^3)(1-x^4)} + \&c$$

If $f(x) = \phi(x) + \phi(2x) + \phi(3x) + \phi(4x) + \&c$

then $\phi(x) + \phi(4x) + \phi(9x) + \phi(16x) + \&c$

$$= f(x) - f(2x) - f(3x) + f(4x) - f(5x) + f(6x) - f(7x)$$

$$- f(8x) + f(9x) + f(10x) - f(11x) - f(12x) - \&c$$

If $2, 3, 5, 7, 11$ &c are prime numbers, then

$$\sqrt{\frac{a(a-1)}{4} + \sqrt{\frac{a(a-1)}{4} + \sqrt{\frac{a(a-1)}{4} + \&c \text{ to } n \text{ terms} + h}}}$$

$$= \frac{a}{2}\left\{1 - \mu/a^n + \frac{(\mu/a^n)^2}{2(a-1)} - \frac{(\mu/a^n)^3}{2(a-1)(a^2-1)} + \frac{(\mu/a^n)^4(a+5)}{8(a-1)(a^2-1)(a^3-1)}\right.$$

$$\left. - \frac{(\mu/a^n)^5(2a^2+3a+7)}{8(a-1)(a^2-1)(a^3-1)(a^4-1)} + \&c\right\}$$

where μ is a function of a & h independent of n.

$$\frac{2h}{a} = 1 - \mu + \frac{\mu^2}{2(a-1)} - \frac{\mu^3}{2(a-1)(a^2-1)} + \&c$$

4. Shew that

i. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \&c = \frac{\pi^2}{6}$

ii. $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \&c = \frac{\pi^3}{32}$

iii. $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \&c = \frac{\pi^4}{90}$

7. $\frac{1}{1-a_2} \cdot \frac{1}{1-a_3} \cdot \frac{1}{1-a_5} \cdot \frac{1}{1-a_7} \cdot \frac{1}{1-a_{11}} \cdot \&c$ where 2, 3, 5, 7, 11, 13 &c are all prime numbers

$= 1 + a_2 + a_3 + a_2 a_2 + a_5 + a_2 a_3 + a_7 + a_2 a_2 a_2 + \&c$ where the suffixes are natural numbers resolved into prime factors

Cor 1. $\frac{2^n}{2^n-1} \cdot \frac{3^n}{3^n-1} \cdot \frac{5^n}{5^n-1} \cdot \frac{7^n}{7^n-1} \cdot \frac{11^n}{11^n-1} \cdot \&c$ where 2, 3, 5, 7 &c are prime numbers $= \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c$ or S_n

Cor 2. $\frac{2^n}{2^n+1} \cdot \frac{3^n}{3^n+1} \cdot \frac{5^n}{5^n+1} \&c = \frac{S_{2n}}{S_n}$

Sol. $\frac{2^n}{2^n-1} \cdot \frac{3^n}{3^n-1} \cdot \frac{5^n}{5^n-1} \&c = S_n$

$\therefore \frac{2^{2n}}{2^{2n}-1} \cdot \frac{3^{2n}}{3^{2n}-1} \cdot \frac{5^{2n}}{5^{2n}-1} \&c = S_{2n}$

$\therefore \frac{2^n}{2^n+1} \cdot \frac{3^n}{3^n+1} \cdot \frac{5^n}{5^n+1} \&c = \frac{S_{2n}}{S_n}$

Cor 3. $\frac{2^n+1}{2^n-1} \cdot \frac{3^n+1}{3^n-1} \cdot \frac{5^n+1}{5^n-1} \&c = \frac{(S_n)^2}{S_{2n}}$

Cor 4. $\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{8^n} + \frac{1}{11^n} + \frac{1}{12^n} + \&c$ where 2, 3, 5, 7, 8 are natural numbers containing an odd number of prime factors

$= \frac{(S_n)^2 - S_{2n}}{2 S_n}$

Sol. Subtract Cor 1 from Cor 1 after applying VII 7.

$$\frac{1}{2\pi\sqrt{3}\,x^4} + \frac{1}{1^4+1^2x^2+x^4} + \frac{2}{2^4+2^2x^2+x^4} + \frac{3}{3^4+3^2x^2+x^4} + \&c$$

$$= \frac{\pi}{3x^2\sqrt{3}} \cdot \frac{\cosh \pi x\sqrt{3} + 2\cos \pi x}{\cosh \pi x\sqrt{3} - \cos \pi x}$$

$$+ 2\left\{ \frac{1}{e^{\pi\sqrt{3}}+1} \cdot \frac{1}{1^4+1^2x^2+x^4} - \frac{2}{e^{2\pi\sqrt{3}}-1} \cdot \frac{1}{2^4+2^2x^2+x^4} + \&c \right\}$$

$$\frac{1}{2n^2} + \frac{1}{1^2+n+n^2} + \frac{1}{2^2+2n+n^2} + \frac{1}{3^2+3n+n^2} + \&c$$

$$+ 2n\left\{ \frac{1}{e^{\pi\sqrt{3}}+1} \cdot \frac{1}{1^4+n^2+n^4} - \frac{2}{e^{2\pi\sqrt{3}}-1} \cdot \frac{1}{2^4+2^2n^2+n^4} + \&c \right\}$$

$$= \frac{1}{2\pi n^3\sqrt{3}} + \frac{2\pi}{3n\sqrt{3}} - \frac{2\pi}{n\sqrt{3}} \cdot \frac{1}{e^{2\pi n\sqrt{3}} - 2e^{\pi n\sqrt{3}}\cos \pi n + 1}$$

$$\frac{1}{6n^2} + \frac{1}{1^2+3n+3n^2} + \frac{1}{2^2+6n+3n^2} + \frac{1}{3^2+9n+3n^2} + \&c.$$

$$+ 6n\left\{ \frac{1}{e^{\pi\sqrt{3}}+1} \cdot \frac{1}{1^4-3n^2+9n^4} - \frac{2}{e^{2\pi\sqrt{3}}-1} \cdot \frac{1}{2^4-2^2.3n^2+9n^4} + \&c \right\}$$

$$= \frac{1}{6\pi n^3\sqrt{3}} + \frac{\pi}{3n\sqrt{3}} - \frac{2\pi}{n\sqrt{3}} \cdot \frac{1}{e^{2\pi n\sqrt{3}} - 2e^{\pi n\sqrt{3}}\cos 3\pi n + 1}$$

Cor 5. $\frac{3^n}{3^n+1} \cdot \frac{5^n}{5^n-1} \cdot \frac{7^n}{7^n+1} \cdot \frac{11^n}{11^n+1} \cdots\cdots \frac{p^n}{p^n - \sin\frac{\pi p}{2}} \cdots$ ad inf.
where p is a prime number
$= \frac{1}{1^n} - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \&c$

8. $(1+a_2)(1+a_3)(1+a_5)(1+a_7)(1+a_{11})$ &c where 2, 3, 5, 7 &c are prime numbers
$= 1 + a_2 + a_3 + a_5 + a_2 a_3 + a_7 + a_2 a_5 + a_{11} + a_{13} + \&c$
where the suffixes are natural numbers resolved into prime factors no two of which are alike

Cor 1. $\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \&c = \frac{S_n}{S_{2n}}$

Cor 2. $\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \frac{1}{19^n}$
$+ \frac{1}{23^n} + \frac{1}{29^n} + \frac{1}{30^n} + \frac{1}{31^n} + \frac{1}{37^n} + \&c = \frac{(S_n)^2 - S_{2n}}{2 S_n S_{2n}}$
where 2, 3, 5, 7 &c are natural numbers containing an odd number of prime factors no two of which are alike.

Cor 3. $\frac{1}{4^n} + \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{12^n} + \&c = \frac{S_n(S_{2n}-1)}{S_{2n}}$
where 4, 8, 9, 12 &c are composite numbers containing at least two equal prime numbers

Ex. 1. Show that the sum of the reciprocals of all prime numbers is infinite.

Sol. From VII 1 Cor 1 we have
$\frac{2}{2-1} \cdot \frac{3}{3-1} \cdot \frac{5}{5-1} \cdot \frac{7}{7-1}$ &c $= 1 + \frac{1}{2} + \frac{1}{3} + \&c$
$\therefore \log_e \frac{2}{2-1} + \log_e \frac{3}{3-1} + \log_e \frac{5}{5-1} + \&c = \log_e\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c\right)$
i.e. $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \&c$ + a finite quantity
$= \infty$ $\therefore$ The sum of the reciprocals of all prime nos $= \infty$

2. Shew that when $n=0$, $\log_e n + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{5^{n+1}} + \&c$ where 2, 3, 5, 7 &c are prime numbers is a finite quantity

Sol. From VII 7 Cor 1 we have

$$\frac{2^{n+1}}{2^{n+1}-1} \cdot \frac{3^{n+1}}{3^{n+1}-1} \cdot \frac{5^{n+1}}{5^{n+1}-1} \cdot \&c = \frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \&c$$

$$\therefore \log_e \frac{2^{n+1}}{2^{n+1}-1} + \log_e \frac{3^{n+1}}{3^{n+1}-1} + \log_e \frac{5^{n+1}}{5^{n+1}-1} + \&c = \log_e\left(\frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \&c\right)$$

$= -\log_e n$ when $n=0$

+.312

$\therefore \log_e n + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{5^{n+1}} + \&c$ is a finite quantity when $n=0$.

3. If 2, 3, 5, 7, 11 are all prime numbers shew that

i. $\frac{2^2+1}{2^2-1} \cdot \frac{3^2+1}{3^2-1} \cdot \frac{5^2+1}{5^2-1} \cdot \frac{7^2+1}{7^2-1} \cdot \frac{11^2+1}{11^2-1} \&c = \frac{5}{2}$

ii. $\frac{2^4+1}{2^4-1} \cdot \frac{3^4+1}{3^4-1} \cdot \frac{5^4+1}{5^4-1} \cdot \frac{7^4+1}{7^4-1} \cdot \frac{11^4+1}{11^4-1} \&c = \frac{7}{6}$

4. If 2, 3, 5, 7, 8 &c are natural numbers ~~are natural numbers~~ containing an odd no. of prime factors shew that

i. $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \&c = \frac{\pi^2}{20}$

ii. $\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{8^4} + \&c = \frac{\pi^4}{1260}$

5. $\frac{1}{2}\sin\frac{2\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \frac{1}{7}\sin\frac{7\pi}{2} + \&c$ is a convergent series, 2, 3, 5, 7 &c being prime nos.

$$\frac{\frac{\log_e 1}{1^n} + \frac{\log_e 2}{2^n} + \frac{\log_e 3}{3^n} + \frac{\log_e 4}{4^n} + \&c}{\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \&c} = \frac{\log_e 2}{2^n-1} + \frac{\log_e 3}{3^n-1} + \frac{\log_e 5}{5^n-1} + \&c$$

where 2, 3, 5, 7, 11 are prime nos.

$$f(1) + f(2) + f(3) + \cdots + f(h)$$

$$= \{f(1) + f(2) + f(3) + \cdots + f(n)\}$$

$$- \{f(1+h) + f(2+h) + \cdots + f(n+h)\}$$

$$+ h f(n) + \frac{\Sigma h}{\lfloor 1} f'(n) + \frac{\Sigma h^2}{\lfloor 2} f''(n) + \&c$$

N. B. When n becomes infinite we may neglect $f^n(n)$ and the terms succeeding $f^2(n)$ if it is 0,

e.g. $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{h}$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

$$- \left(\frac{1}{1+h} + \frac{1}{2+h} + \frac{1}{3+h} + \cdots + \frac{1}{n+h}\right) \text{ when } n = \infty.$$

$$= c_0 + \log_e n - \left(\frac{1}{1+h} + \frac{1}{2+h} + \frac{1}{3+h} + \cdots \frac{1}{n+h}\right) \text{ when } n = \infty$$

$$\lfloor h = \frac{n^h}{(1+\frac{h}{1})(1+\frac{h}{2})(1+\frac{h}{3}) \cdots (1+\frac{h}{n})} \text{ when } n = \infty.$$

$$f(1) + f(2) + f(3) + \cdots + f(h)$$

$$= x f(1) - x^{1+h} f(1+h) + x^2 f(2) - x^{2+h} f(2+h) + \&c$$

when x becomes unity.

CHAPTER VIII

1. $f(1) + f(2) + f(3) + f(4) + \cdots\cdots\cdots + f(x) = \phi(x).$

$$\phi(x) = C + \int f(x)\,dx + \frac{1}{2}f(x) + \frac{B_2}{\lfloor 2}f'(x) - \frac{B_4}{\lfloor 4}f'''(x) + \frac{B_6}{\lfloor 6}f^{V}(x) - \frac{B_8}{\lfloor 8}f^{VII}(x) + \&c$$

Sol. $\phi(x) - \phi(x-1) = f(x)$; apply VI 1.

N. B. 1. By giving any value to x, C can be found. R.H.S is not a terminating series except in some special cases. Consequently no constant can be found in $\frac{1}{2}f(x) + \frac{B_2}{\lfloor 2}f'(x) - \frac{B_4}{\lfloor 4}f'''(x) + \&c$ except in those special cases. If R.H.S be a terminating series, it must be some integral function of x; in this case there is no possibility of a constant (according to the ordinary sense) in $\phi(x)$; for, $\phi(1) = f(1) + \phi(0)$. But $\phi(1) = f(1)$ $\therefore \phi(0)$ is always 0 whether $\phi(x)$ is rational or irrational. $\therefore$ When $\phi(x)$ is a rational integral function of (x) it must be divisible by x since $\phi(0) = 0$ and consequently no constant but 0 can exist. Therefore let us try to give some other meaning for the constant of a series.

The constant of a series is the constant obtained by completing the remaining part by faithfully adhering to the above rule for summation of series. The constant of a series has some mysterious connection with the given infinite series and it is like the centre of gravity of a body. Mysterious because we may substitute it for the divergent infinite series. Now the constant of the series $1+1+1+1+\&c$ is $-\frac{1}{2}$; for the sum to x terms $= x = C + \int 1\,dx + \frac{1}{2}$. $\therefore C = -\frac{1}{2}$.

$\sqrt{21}$ $\quad \frac{1}{2}\left(\frac{3-\sqrt{7}}{\sqrt{2}}\right)^2 \left(\sqrt{\frac{5+\sqrt{7}}{4}} - \sqrt{\frac{1+\sqrt{7}}{4}}\right)^4 \left(\sqrt{\frac{3+\sqrt{7}}{4}} - \sqrt{\frac{\sqrt{7}-1}{4}}\right)^4 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3.$

$\sqrt{33}$ $\quad \frac{1}{2}(2-\sqrt{3})^3 \left(\sqrt{\frac{7+3\sqrt{3}}{4}} - \sqrt{\frac{3+3\sqrt{3}}{4}}\right)^4 \left(\sqrt{\frac{5+\sqrt{3}}{4}} - \sqrt{\frac{1+\sqrt{3}}{4}}\right)^4 \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^2.$

$\sqrt{45}$ $\quad \frac{1}{2}(\sqrt{5}-2)^3 \left(\sqrt{\frac{7+3\sqrt{5}}{4}} - \sqrt{\frac{3+3\sqrt{5}}{4}}\right)^4 \left(\sqrt{\frac{3+\sqrt{5}}{2}} - \sqrt{\frac{1+\sqrt{5}}{2}}\right)^4 \left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^4.$

Another way of finding the constant is as follows — Let us take the series $1+2+3+4+5+\&c$. Let C be its constant. Then

$$C = 1+2+3+4+\&c$$
$$\therefore 4C = \quad 4 \quad +8 \quad +\&c$$
$$\therefore -3C = 1-2+3-4+\&c = \frac{1}{(1+1)^2} = \frac{1}{4}$$

$\therefore C = -\frac{1}{12}$.

N.B 2. For finding the sum to a fractional number of terms assume the sum to be true always and if there is anything difficult in finding $\phi(h)$ where h is small, take n any integer you choose, find $\phi(n+h)$ and then subtract $\{f(1+h)+f(2+h)+\cdots+f(n+h)\}$ from the result.

The sum to a negative number of terms is the sum with the sign changed, calculated backwards from the term previous to the first to the given no. of terms with positive sign instead of negative.

Cor. $\phi(x) = \sum_{n=0}^{n=\infty} \frac{B_n}{\lfloor n} \overset{n-1}{f}(x) \cos\frac{\pi x}{2}$

Sol. Let $\frac{B_n}{\lfloor n}\psi(n)$ be the coeff^t. of $\overset{n-1}{f}(x)$, then we see

$\psi(0)=1,\ \psi(2)=-1,\ \psi(4)=1,\ \psi(6)=-1\ \&c$

$\psi(3)=0,\ \psi(5)=0,\ \psi(7)=0\ \&c.\quad \frac{B_1}{\lfloor 1}\psi(1)=\frac{1}{2}$ But $B_1=\infty$

$\therefore \psi(1)=0$. Again by III 6 ex 2 we have $\pi(n-1)B_n=1$ when $n=1$ $\therefore \frac{B_n\psi(n)}{\lfloor n} = \frac{\pi(n-1)B_n}{\lfloor n}\cdot\frac{\psi(n)}{\pi(n-1)} = \frac{1}{2}$ when $n=1$, i.e. $\frac{\psi(n)}{\pi(n-1)} = \frac{1}{2}$ when $n=1$. $\therefore \psi(n) = -\cos\frac{\pi n}{2}$.

$$\therefore \phi(x) + \sum_{n=0}^{n=\infty}\left\{\frac{B_n}{\lfloor n}\overset{n-1}{f}(x)\cos\frac{\pi n}{2}\right\} = 0.$$

2	$(\sqrt{2}-1)^4$
$2\sqrt{3}$	$(\sqrt{3}-\sqrt{2})^4(\sqrt{2}-1)^4$
$2\sqrt{7}$	$(\sqrt{2}-1)^8(2\sqrt{2}-\sqrt{7})^4$
$2\sqrt{15}$	$(\sqrt{10}-3)^4(\sqrt{6}-\sqrt{5})^4(\sqrt{3}-\sqrt{2})^4(\sqrt{2}-1)^4.$
$2\sqrt{2}$	$\left(\sqrt{3+2\sqrt{2}}-\sqrt{2+2\sqrt{2}}\right)^4$
$2\sqrt{6}$	$\left(\sqrt{6+3\sqrt{3}}-\sqrt{5+3\sqrt{3}}\right)^4\left(\sqrt{2+\sqrt{3}}-\sqrt{1+\sqrt{3}}\right)^4$
$2\sqrt{10}$	$\left(2\sqrt{2}+\sqrt{5}-2\sqrt{3+\sqrt{10}}\right)^4\left(\sqrt{2}+\sqrt{5}-\sqrt{6+2\sqrt{10}}\right)^4.$

If $e^{-\pi\sqrt{2p}} = F\,(\sqrt{n+1}-\sqrt{n})^2(\sqrt{n}-\sqrt{n-1})^2$

then $e^{-2\pi\sqrt{2p}} = F\left\{\frac{\sqrt{n}+1+\sqrt{n+1}}{\sqrt{2}}-\sqrt{(\sqrt{n}+1)(\sqrt{n}+\sqrt{n+1})}\right\}^4$

$\times\left\{\frac{\sqrt{n}-1+\sqrt{n+1}}{\sqrt{2}}-\sqrt{(\sqrt{n}-1)(\sqrt{n}+\sqrt{n+1})}\right\}^4$

If $e^{-\pi\sqrt{p}} = F\,\frac{1-\sqrt{1-\frac{1}{n^2}}}{2}$

then $e^{-2\pi\sqrt{p}} = F\,(\sqrt{n+1}-\sqrt{n})^4(\sqrt{n}-\sqrt{n-1})^4,$

& $e^{-4\pi\sqrt{p}} = F\left\{(\sqrt{n+1}+\sqrt{n})(\sqrt{2n}+1)-\sqrt{\cdots}\right\}^4$

$\times\left\{(\sqrt{n+1}+\sqrt{n})(\sqrt{2n}-1)-\sqrt{\cdots}\right\}^4$

$= F\,(\sqrt{n+1}+\sqrt{n})^8\left\{\sqrt{2n}+1-\sqrt{2\sqrt{n}(\sqrt{n+1}+\sqrt{2})}\right\}^4$

$\times\left\{\sqrt{2n}-1-\sqrt{2\sqrt{n}(\sqrt{n+1}-\sqrt{2})}\right\}^4.$

2. Def. A series is said to be *corrected* when it's constant is subtracted from it.

Theorem. – The differential coeff[t] of a series is a corrected series.

i.e $\frac{d\{\phi(1)+\phi(2)+\phi(3)+\cdots+\phi(x)\}}{dx} = \phi'(1)+\phi'(2)+\cdots+\phi'(x) - c'$

where c' is is the constant of $\phi'(1)+\phi'(2)+\phi'(3)+\cdots+\phi'(x)$

Sol. In the differential coeff[t] of $\phi(1)+\phi(2)+\cdots+\phi(x)$ there can't be any constant. Therefore it should be corrected.

N.B. If $f(1)+f(2)+\cdots+f(x)+\&c$ be a convergent series then it's constant is the sum of the series itself.

Ex. 1 $\frac{d\,\Sigma\frac{1}{x}}{dx} = \frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\&c$

Sol. $\frac{d\,\Sigma\frac{1}{x}}{dx} = -\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}-\cdots-\frac{1}{x^2}-c$ by VIII 2 N.B.

$= \frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\&c$

2. If S_1 be the constant of $\Sigma\frac{1}{x}$, then $\frac{d\lfloor x}{dx} = \lfloor x\left(\Sigma\frac{1}{x}-S_1\right)$

Sol. $\frac{d\lfloor x}{dx} = \lfloor x \,\frac{d\log_e\lfloor x}{dx} = \lfloor x\left(\Sigma\frac{1}{x}-S_1\right)$.

3. $\int_0^x \Sigma\frac{1}{x}\,dx = \log_e\lfloor x + S_1 x$

4. $\int_0^x (1^{13}+2^{13}+3^{13}+\cdots x^{13})\,dx = \frac{1}{14}(1^{14}+2^{14}+\cdots+x^{14}) - \frac{x}{12}$

5. $\frac{d(1^{10}+2^{10}+3^{10}+\cdots+x^{10})}{dx} = 10\left(1^9+2^9+\cdots+x^9 + \frac{1}{132}\right)$.

6. $\int_0^x (\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{x})\,dx = \frac{2}{3}(1\sqrt{1}+2\sqrt{2}+3\sqrt{3}+\cdots+x\sqrt{x})$

$- \frac{x}{4\pi}\left(\frac{1}{1\sqrt{1}}+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\&c\right)$

If $S_n = \frac{B_n}{2n} + (-1)^{\frac{n}{2}} \left\{ \frac{1^{n-1} \not{x}}{x-1} + \frac{2^{n-1}}{x^2-1} + \frac{3^{n-1}}{x^3-1} + \&c \right\}$

then $S_{n+2} \frac{(n+2)(n+3)}{2} + \frac{n(n-1)(n-2)(n-3)}{\underline{|2}} S_4 S_{n-2} (n-2)(n-13)$

$+ \frac{n \ldots (n-5)}{\underline{|4}} S_6 S_{n-4} (n-7)(n-18)$ ~~n is even & g~~

$+ \frac{n \ldots (n-7)}{\underline{|6}} S_8 S_{n-6} \left\{ (n-12)(n-23) - 5.1.6 \right\}$

$+ \frac{n \ldots (n-9)}{\underline{|8}} S_{10} S_{n-8} \left\{ (n-17)(n-28) - 5.2.7 \right\}$

$+ \frac{n \ldots (n-11)}{\underline{|10}} S_{12} S_{n-10} \left\{ (n-22)(n-33) - 5.3.8 \right\} + \&c$

If the last term be a perfect Sq then take ½ the terms only.

n is any even no. greater than 6. $\underline{S_8 = 120 S_4^2}$

3. If $f^{n}(x)$ stands for the nth derivative of $f(x)$ and c_n be the constant of $\{f^{n}(1) + f^{n}(2) + \cdots + f^{n}(x)\}$, then

$$\phi(x) = -c_1 x - c_2 \frac{x^2}{\lfloor 2} - c_3 \frac{x^3}{\lfloor 3} - c_4 \frac{x^4}{\lfloor 4} - \&c$$

Sol. We know $\phi(x) = \phi(0) + \frac{x}{\lfloor 1}\phi'(0) + \frac{x^2}{\lfloor 2}\phi''(0) + \&c$

From VIII 2 we have $\phi(0) = 0$, $\phi'(0) = -c_1$, $\phi''(0) = -c_2$ &c

Ex. 1. Shew that $\log_e \lfloor x = -S_1 x + \frac{S_2}{2}x^2 - \frac{S_3}{3}x^3 + \frac{S_4}{4}x^4 - \&c$ where S_n is the constant of $\left(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c\right)$

2. $\Sigma \frac{1}{x} = S_2 x - S_3 x^2 + S_4 x^3 - S_5 x^4 + \&c$ where $S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c$

N.B. This is very useful in finding $\phi(x)$ for fractional values of x.

4. If c'_n be the constant of $f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \cdots + f\left(\frac{x}{n}\right)$, then

$$\phi\left(\frac{x}{n}\right) + \phi\left(\frac{x-1}{n}\right) + \phi\left(\frac{x-2}{n}\right) + \cdots + \phi\left(\frac{x-n+1}{n}\right) - nc$$
$$= f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \cdots + f\left(\frac{x}{n}\right) - c'_n$$

Sol. Let $\psi(x) = \phi\left(\frac{x}{n}\right) + \phi\left(\frac{x-1}{n}\right) + \cdots + \phi\left(\frac{x-n+1}{n}\right)$ then

$$\psi(x) - \psi(x-1) = \phi\left(\frac{x}{n}\right) - \phi\left(\frac{x-n}{n}\right) = f\left(\frac{x}{n}\right)$$

$\therefore$ $\psi(x)$ and $f\left(\frac{x}{n}\right) + f\left(\frac{x-1}{n}\right) + \cdots f\left(\frac{1}{n}\right)$ differ only by some constant; hence if these be corrected they must be equal. $\psi(x)$ contains n terms each of which is of the form $\phi(y)$ whose constant is c. $\therefore$ The constant of $\psi(x) = nc$. And the constant of $f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \cdots + f\left(\frac{x}{n}\right)$ is c'_n by our supposition.

Cor. $\phi\left(-\frac{1}{n}\right) + \phi\left(-\frac{2}{n}\right) + \phi\left(-\frac{3}{n}\right) + \cdots + \phi\left(-\frac{n-1}{n}\right) = nc - c'_n$

Sol. Put $x = 0$ in the above theorem

Ex. 1 Shew that $\phi\left(-\frac{1}{2}\right) = 2c - c'_2$.

2 $c = c_0 - c'_1$.

$\sqrt[48]{256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)} = P$ & $\sqrt[48]{\dfrac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} = Q$

1, 3, 5, 15 :— $Q^3 + \frac{1}{Q^3} - \sqrt{2}\left(\frac{1}{P} + P\right) = 0.$

1, 3, 7, 21 :— $\left(Q^{16} + \frac{1}{Q^{16}}\right) - 5\left(Q^{12} + \frac{1}{Q^{12}}\right) + 5\left(Q^8 + \frac{1}{Q^8}\right) + 6\left(Q^4 + \frac{1}{Q^4}\right)$
$- \left(\frac{8}{P^6} - 6 + 8P^6\right) = 0.$

1, 5, 7, 35 :— $Q^4 + \frac{1}{Q^4} - \left(Q^2 + \frac{1}{Q^2}\right) - 2\left(P^2 + \frac{1}{P^2}\right) = 0.$

~~1, 5, 11, 55~~ :— { 3, 1, 5, 15 :— $\left(Q^4 + \frac{1}{Q^4}\right) - 2\left(P^2 + \frac{1}{P^2}\right) + 3 = 0.$
5, 1, 3, 15 :— $\left(Q^6 + \frac{1}{Q^6}\right) - 4\left(P^4 + \frac{1}{P^4}\right) + 10\left(P^2 + \frac{1}{P^2} - 1\right) = 0$

7, 1, 3, 21 $\left(Q^8 + \frac{1}{Q^8}\right) + 7\left(Q^6 + \frac{1}{Q^6}\right) + 14\left(Q^4 + \frac{1}{Q^4}\right) + 21\left(Q^2 + \frac{1}{Q^2}\right) + 42$
$- 8\left(P^6 + \frac{1}{P^6}\right) = 0$

5, 1, 7, 35 $\left(Q^6 + \frac{1}{Q^6}\right) + 5\sqrt{2}\left(Q^3 + \frac{1}{Q^3}\right)\left(P + \frac{1}{P}\right) + 10$
$- 4\left(P^4 + \frac{1}{P^4}\right) = 0.$

5, 1, 11, 55 $\left(Q^6 + \frac{1}{Q^6}\right) - 5\left(Q^4 + \frac{1}{Q^4}\right) + 10\left(Q^2 + \frac{1}{Q^2}\right)\left(\frac{1}{P^2} - 1 + P^2\right)$
$- \left(\frac{4}{P^4} - \frac{10}{P^2} + 25 - 10P^2 + 4P^4\right) = 0.$

3, 1, 11, 33 $Q^4 + \frac{1}{Q^4} + 3\left(Q^2 + \frac{1}{Q^2}\right) - 2\left(P^2 + \frac{1}{P^2}\right) = 0$

$\sqrt[48]{\dfrac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}} = P$ & $\sqrt[48]{\dfrac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} = Q$

1, 5, 7, 35 $\left(P^4 + \frac{1}{P^4}\right) - \left(Q^6 + \frac{1}{Q^6}\right) + 5\left(Q^4 + \frac{1}{Q^4}\right) - 10\left(Q^2 + \frac{1}{Q^2}\right) + 15 = 0$

3. $\phi(-\frac{1}{3}) + \phi(-\frac{2}{3}) = 3c - c'_3$

4. $\phi(-\frac{1}{4}) + \phi(-\frac{3}{4}) = 2c + c'_2 - c'_4$

5. $\phi(-\frac{1}{6}) + \phi(-\frac{5}{6}) = c + c'_2 + c'_3 - c'_6$

6. $\phi(-\frac{1}{2}) = c + \int f(x)\,dx - \frac{2-1}{2}\frac{B_2}{\lfloor 2} f'(x) + \frac{2^3-1}{2^3}\frac{B_4}{\lfloor 4} f'''(x) - \&c$

$$= \sum_{n=0}^{n=\infty} \left\{ \left(1 - \frac{1}{2^{n-1}}\right) \frac{B_n}{\lfloor n} f^{n-1}(x) \cos\frac{\pi n}{2} \right\}$$

5. $S(a_1 + a_2 + a_3 + \&c)$ means that the series is a convergen[t] series and its sum to infinity is req^d

II $C(a_1 + a_2 + a_3 + \&c)$ means that the series is a pure divergent series and its constant is req^d.

III. $G(a_1 + a_2 + a_3 + \&c)$ means that the series is an oscillat[ing] series (convergent or divergent) or that the series is a pure divergent series whose sum to n terms can not be [illegible] and consequently its constant also and that in both cases the value of the generating function is required.

N.B. Here-after the series will only be given omitting S, C or G and from the nature of the series we [illegible] in for whether C, S or G is req^d. Moreover if a series appear to be equal to a finite quantity we must select S, C or G from the nature of the series.

IV. The value of [illegible] series is only true when the series is deduced from a regular series. For example the value of $1 - 1 + 1 - 1 + 1 - 1 \&c = \frac{1}{2}$ only when it is deduced from a [illegible] series of the form $\{\phi(1)\}^c - \{\phi(2)\}^c + \{\phi(3)\}^c - \{\phi(4)\}^c + \&c$ [illegible] series a^c [illegible]

$a_1 - a_2 + a_3 - a_4 + \&c$

$= \frac{a_1}{2} + \frac{a_1 - a_2}{4} + \frac{a_1 - 2a_2 + a_3}{8} + \&c$

$a_1 - a_2 + a_3 - a_4 + \&c$

$= x a_1 - x^2 a_2 + x^3 a_3 - x^4 a_4 + \&c$
$= x\frac{a_1}{2} + x^2 \frac{a_1 - a_2}{4} + x^3 \frac{a_1 - 2a_2 + a_3}{8} + \&c$ } when x approaches unity.

$\sqrt[48]{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}} = P$ $\sqrt[48]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} = Q$

1, 13, 5, 65 : $-4\left(Q^6 + \frac{4}{Q^6}\right) - 5\left(Q^2 + \frac{4}{Q^4}\right)^2\left(P + \frac{4}{P}\right)^2$
$- \left(\frac{4}{P^4} + P^4\right) = 0.$

1, 13, 3, 39 : $-\left(Q^4 + \frac{4}{Q^4}\right) - 3\left(Q^2 + \frac{4}{Q^2}\right) + 3 - \left(P^2 + \frac{4}{P^2}\right) = 0.$

~~1, 3, 11, 33~~ : —

we get the same series $1-1+1-1+\&c$ when n becomes 0
yet its value is not $\frac{1}{2}$ in this case

V. $a_1 - a_2 + a_3 - a_4 + \&c$ is not equal to the series
$(a_1-a_2)+(a_3-a_4)+(a_5-a_6)+\&c$ or to the series
$a-(a_2-a_3)-(a_4-a_5)-(a_6-a_7)-\&c$. For example
$1-2+3-4+5-6+\&c$ is not equal to the Series
$(1-2)+(3-4)+(5-6)+\&c$ or to the Series $1-(2-3)-(4-5)-\&c$.
But $a_1-a_2+a_3-a_4+\&c = a_1-(a_2-a_3+a_4-a_5+\&c)$

VI. $(a_1-a_2+a_3-a_4+\&c)+(b_1-b_2+b_3-b_4+\&c)$
$= (a_1+b_1)-(a_2+b_2)+(a_3+b_3)-(a_4+b_4)+\&c$

Cor. $(a_1-a_2+a_3-a_4+\&c)+(b_1-b_2+b_3-b_4+\&c)$
$= a_1+(b_1-a_2)-(b_2-a_3)+(b_3-a_4)-(b_4-a_5)+\&c$

Sol. From VIII s.1 we know
$a_1-a_2+a_3-a_4+\&c = a_1-(a_2-a_3+a_4-\&c)$
$\therefore (a_1-a_2+a_3-a_4+\&c)+(b_1-b_2+b_3-b_4+\&c)$
$= a_1+(b_1-b_2+b_3-b_4+\&c)+(-a_2+a_3-a_4+\&c)$
$= a_1+(b_1-a_2)-(b_2-a_3)+(b_3-a_4)-\&c$

Ex. 1. $a_1-a_2+a_3-a_4+\&c = \frac{a_1}{2}+\frac{1}{2}\{(a_1-a_2)-(a_2-a_3)+(a_3-a_4)-\&c\}$

2. $a_1-a_2+a_3-a_4+\&c = \frac{3a_1-a_2}{4}+\frac{1}{4}\{(a_1-2a_2+a_3)-(a_2-2a_3+a_4)+(a_3-2a_4+a_5)-\&c\}$

3. $a_1-a_2+a_3-a_4+\&c = \frac{7a_1-4a_2+a_3}{8}+$
$\frac{1}{8}\{(a_1-3a_2+3a_3-a_4)-(a_2-3a_3+3a_4-a_5)+(a_3-\ldots)-\&c$

$$\sqrt[24]{16\alpha\beta(1-\alpha)(1-\beta)} = P \text{ and } \sqrt[24]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} = Q.$$

3. $Q^6 + \frac{1}{Q^6} = 2\left(\frac{1}{P^3} - P^3\right) = 0$

5. $Q^3 + \frac{1}{Q^3} = 2\left(\frac{1}{P^2} - P^2\right) = 0$

7. $Q^4 + \frac{1}{Q^4} - 2\left(\frac{\sqrt{2}}{P^3} - \frac{7}{2} + \sqrt{2}P^3\right) = 0.$

11. $Q^6 + \frac{1}{Q^6} - 2\sqrt{2}\left(\frac{2}{P^5} - \frac{11}{P^3} + \frac{22}{P} - 22P + 11P^3 - 2P^5\right) = 0$

13. $Q^7 + \frac{1}{Q^7} + 13\left(Q^5 + \frac{1}{Q^5}\right) + 52\left(Q^3 + \frac{1}{Q^3}\right) + 78\left(Q + \frac{1}{Q}\right)$

$$- 8\left(\frac{1}{P^6} - P^6\right) = 0.$$

17. $Q^9 + \frac{1}{Q^9} - 34\left(Q^6 + \frac{1}{Q^6}\right) + 17\left(Q^3 + \frac{1}{Q^3}\right)\left(\frac{4}{P^4} + 7 + 4P^4\right)$

$$- \left(\frac{16}{P^8} - \frac{136}{P^4} - 340 - 136P^4 + 16P^8\right) = 0.$$

19. $Q^{10} + \frac{1}{Q^{10}} + 114\left(Q^6 + \frac{1}{Q^6}\right) - 190\sqrt{2}\left(Q^4 + \frac{1}{Q^4}\right)\left(\frac{1}{P^3} - P^3\right)$

$$+ 19\left(Q^2 + \frac{1}{Q^2}\right)\left(\frac{8}{P^6} - 5 + 8P^6\right)$$

$$- 4\sqrt{2}\left(\frac{4}{P^9} + \frac{19}{P^3} - 19P^3 - 4P^9\right) = 0.$$

These are true even for even functions though the signs are changed in most cases.

VII. If $\frac{a_2}{a_3}$ lies between $\frac{a_1}{a_2}$ and $\frac{a_3}{a_4}$, then

$a_1 - a_2 + a_3 - a_4 + \&c$ lies between $\frac{a_1}{a_1+a_2}$ and $a_1 - \frac{a_2^2}{a_2+a_3}$

E.g. $1-2+3-4+\&c$ lies between $\frac{1}{3}$ and $\frac{1}{5}$ and its value is $\frac{1}{4}$. $\lfloor 0 - \lfloor 1 + \lfloor 2 - \lfloor 3 + \&c$ lies between $\frac{1}{2}$ and $\frac{2}{3}$; its value is $\frac{3}{5}$ very nearly.

But $2 - 2\frac{1}{2} + 3\frac{1}{3} - 4\frac{1}{4} + 5\frac{1}{5} - \&c$ cannot lie between $\frac{2^2}{2+2\frac{1}{2}}$ and $2 - \frac{(2\frac{1}{2})^2}{2\frac{1}{2}+3\frac{1}{3}}$ as $\frac{2\frac{1}{2}}{3\frac{1}{3}}$ is not lying between $\frac{2}{2\frac{1}{2}}$ and $\frac{3\frac{1}{3}}{4\frac{1}{4}}$

6. $\phi_1(x) + \phi_2(x) + \phi_3(x) + \&c$ can be expanded in ascending powers of x, say $A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c$ where each of A_0, A_1, A_2 &c is a series.

Case I When A_n is a convergent series.

(1) If $A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c$ be a rapidly convergent series what is required is got.

(2) But if it is a slowly convergent or an oscillating series, convergent or divergent (at least for some values of x)

(a) change x into a suitable function of y so that the new series in ascending powers of y may be a rapidly convergent series; E.g. let [illegible] then

$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c = y + \frac{y^3}{12} + \frac{y^5}{80} + \frac{y^7}{448} + \&c$

(b) or convert it into a continued power [illegible]

E.g. $x - \frac{x^2}{3} + \frac{2x^3}{[illegible]} - \frac{17x^4}{[illegible]} + \&c$

$$\sqrt[8]{\alpha\beta} = P \qquad \sqrt[8]{\frac{\beta}{\alpha}} = Q$$

3. $$\frac{1}{Q^2} - Q^2 - 2\left(\frac{1}{P} - P\right) = 0$$

5. $$\frac{1}{Q^3} - Q^3 + 5\left(\frac{1}{Q} - Q\right) - 4\left(\frac{1}{P^2} - P^2\right) = 0$$

7. $$\frac{1}{Q^4} + Q^4 - 8\left(\frac{1}{P^3} + P^3\right) + 28\left(\frac{1}{P^2} + P^2\right) - 56\left(\frac{1}{P} + P\right) + 70 = 0$$

$$\cfrac{x}{1+\cfrac{x}{3+\cfrac{x}{5+\cfrac{x}{7+\&c}}}} \qquad \frac{\lfloor 0}{x} - \frac{\lfloor 1}{x^2} + \frac{\lfloor 2}{x^3} - \frac{\lfloor 3}{x^3} + \&c = \cfrac{1}{x+1-\cfrac{1^2}{x+3-\cfrac{2^2}{x+5-\&c}}}$$

(c) or transform it into another series by applying III 7 ex. 1

Eg. $\frac{1}{x} - \frac{2}{x^2} + \frac{5}{x^3} - \frac{15}{x^4} + \frac{52}{x^5} - \&c$

$$= \frac{1}{x+1} - \frac{1}{(x+1)(x+2)} + \frac{1}{(x+1)(x+2)(x+3)} - \&c$$

(d) or take the reciprocal of the series and try to make it a rapidly convergent series in any way

Case II When A_n is an oscillating (convergent or divergent) or a pure divergent series.

(1) Let C_n be the constant or the value of its generating function according as it is purely divergent or not. Then the given series $= \psi(x) + C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \&c$ where $\psi(x)$ is a simple function of x.

(2) But if unfortunately $C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \&c$ be a divergent series, find some function of n (say P_n) such that the value of $P_0 + P_1 x + P_2 x^2 + \&c$ may easily found and $C_n - P_n$ may rapidly diminish as n increases. Then the given series $= F(x) + (C_0 - P_0) + (C_1 - P_1)x + (C_2 - P_2)x^2 + (C_3 - P_3)x^3 + \&c$

Eg. 1. $\frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} - \&c = \frac{1}{x}(1 - 1 + 1 - 1 + \&c) - \frac{1}{x^2}(1 - 2 + \&c)$

$$= \frac{1}{2x} - \frac{1}{4x^2} + \&c$$

2. $-\frac{1}{1^2 - x^2} + \frac{1}{2^2 - x^2} \ldots \frac{1}{3^2 - x^2} + \&c = -\frac{1}{x^2}(1 + 1 + 1 + \&c)$

$$- \frac{1}{x^4}(1^2 + 2^2 + 3^2 + \&c) - \frac{1}{x^6}(1^4 + 2^4 + 3^4 + \&c) = \frac{1}{2x^2} + \psi(x)$$

$$= \frac{1}{2x^2} - \frac{\pi \cot \pi x}{2x}$$

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^{2n} dx = \frac{\pi n}{(2n-1)(n+1)}\left(1+\frac{1}{2}+\cdots\frac{1}{n}\right).$$

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^{2n+1} dx = 2\pi \cdot \frac{1^2\cdot 3^2 \cdots \overline{2n-1}^2}{2^2\cdot 4^2 \cdots 2n^2}\left(1+\frac{1}{3}+\cdots\frac{1}{2n-1}\right),$$

$$1+\frac{\frac{1}{3}\cdot\frac{2}{3}}{1\cdot\frac{1}{2}}x+\frac{\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{5}{3}}{1\cdot 2\cdot\frac{1}{2}\cdot 1\frac{1}{2}}x^2+\&c$$

$$= \frac{\cos\left(\frac{1}{3}\sin^{-1}\sqrt{x}\right)}{\sqrt{1-x}}.$$

$$1-\frac{\frac{1}{3}\cdot\frac{2}{3}}{1\cdot\frac{1}{2}}x+\frac{\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{5}{3}}{1\cdot 2\cdot\frac{1}{2}\cdot 1\frac{1}{2}}x^2-\&c$$

$$= \frac{\sqrt[3]{\sqrt{1+x}+\sqrt{x}}+\sqrt[3]{\sqrt{1+x}-\sqrt{x}}}{2\sqrt{1+x}}.$$

$$1+\frac{\frac{1}{3}\cdot\frac{2}{3}}{1\cdot 1\frac{1}{2}}x+\frac{\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{5}{3}}{1\cdot 2\cdot 1\frac{1}{2}\cdot 2\frac{1}{2}}x^2+\&c$$

$$= \frac{3}{\sqrt{x}}\sin\left(\frac{1}{3}\sin^{-1}\sqrt{x}\right)$$

$$1-\frac{\frac{1}{3}\cdot\frac{2}{3}}{1\cdot 1\frac{1}{2}}x+\frac{\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{5}{3}}{1\cdot 2\cdot 1\frac{1}{2}\cdot 2\frac{1}{2}}x^2-\&c$$

$$= \frac{3}{2\sqrt{x}}\left(\sqrt[3]{\sqrt{1+x}+\sqrt{x}}-\sqrt[3]{\sqrt{1+x}-\sqrt{x}}\right)$$

$$\cos\left(2n\sin^{-1}\sqrt{x}\right),\quad \frac{\sin\left(2n\sin^{-1}\sqrt{x}\right)}{2n\sqrt{x}},\quad \frac{\cos 2n\sin^{-1}\sqrt{x}}{\sqrt{1-x}}$$

$$\frac{\sin\left(2n\sin^{-1}\sqrt{x}\right)}{2n\sqrt{x(1-x)}}$$

3. $\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \&c = (1+1+1+\&c)$

$- x(\log_e 1 + \log_e 2 + \log_e 3 + \&c) + \&c = -\frac{1}{2} - x\log_e\sqrt{2\pi} - \&c$

$= \frac{1}{x-1} + 1 + x + x^2 + \&c - \frac{1}{2} - x\log_e\sqrt{2\pi} - \&c$

$= \frac{1}{x-1} + \frac{1}{2} + x(1 - \log_e\sqrt{2\pi}) - \&c.$

$= \frac{1}{x-1} + \frac{1}{2} + (1 - .91894)x - \&c = \frac{1}{x-1} + .5 + .08106x - \&c$

Cor. 1. $\frac{x}{e^x+1} + \frac{x}{e^{2x}+1} + \frac{x}{e^{3x}+1} + \frac{x}{e^{4x}+1} + \&c$

$= \log_e 2 - \frac{x}{4} + (B_2)^2 \frac{x^2(2^2-1)}{2\,\underline{|2}} + (B_4)^2 \frac{x^4(2^4-1)}{4\,\underline{|4}} + (B_6)^2 \frac{x^6(2^6-1)}{6\,\underline{|6}} + \&c.$

Sol. By chapter VI 5 cor $\frac{x}{e^{nx}+1} = \frac{x}{2} - B_2 \frac{n\,x^2(2^2-1)}{\underline{|2}}$

$+ B_4 \frac{n^3 x^4(2^4-1)}{\underline{|4}} - B_6 \frac{n^5 x^6(2^6-1)}{\underline{|6}} + \&c$

$\therefore \frac{x}{e^x+1} + \frac{x}{e^{2x}+1} + \frac{x}{e^{3x}+1} + \&c$

$= \frac{x}{2}(1+1+1+\&c) - B_2 \frac{x^2(2^2-1)}{\underline{|2}}(1+2+3+\&c)$

$+ B_4 \frac{x^4(2^4-1)}{\underline{|4}}(1^3+2^3+3^3+\&c) - \&c$

$= \psi(x) - \frac{x}{4} + (B_2)^2 \frac{x^2(2^2-1)}{2\,\underline{|2}} + (B_4)^2 \frac{x^4(2^4-1)}{4\,\underline{|4}} + \&c.$

Now it is reqd. to find ψx.

The given series $= \frac{x}{e^x-1} - \frac{x}{e^{2x}-1} + \frac{x}{e^{3x}-1} - \&c$

Here the term excepting x and higher powers of x is $\log_e 2$ $\therefore \psi(x) = \log_e 2$.

Cor 2. $\frac{x}{e^x-1} + \frac{x}{e^{2x}-1} + \frac{x}{e^{3x}-1} + \frac{x}{e^{4x}-1} - \&c$

$= S_1 - \log_e x + \frac{x}{4} - (B_2)^2 \frac{x^2}{2\,\underline{|2}} - (B_4)^2 \frac{x^4}{4\,\underline{|4}} - (B_6)^2 \frac{x^6}{6\,\underline{|6}} - \&c$

Sol. Proceeding as in the previous theorem we have

the series $= \psi(x) + S_1 + \frac{x}{4} - (B_2)^2 \frac{x^2}{2\,\underline{|2}} - (B_4)^2 \frac{x^4}{4\,\underline{|4}} - \&c$

$$1+6\left(\frac{1}{e^{y}+e^{-y}+1}+\frac{-1}{e^{2y}+e^{-2y}+1}+\&c\right)$$

$$=1+\frac{1\cdot 2}{3^2}x+\&c=Z$$

$$\frac{1^2}{e^{y}+e^{-y}+1}+\frac{2^2}{e^{2y}+e^{-2y}+1}+\&c=\frac{x}{27}Z^3$$

$$\frac{1^4}{e^{y}+e^{-y}+1}+\frac{2^4}{e^{2y}+e^{-2y}+1}+\&c=\frac{x}{27}Z^5$$

$$\frac{1^6}{e^{y}+e^{-y}+1}+\frac{2^6}{e^{2y}+e^{-2y}+1}+\&c=\frac{x}{27}\left(1+\frac{4x}{3}\right)Z^7$$

$$\frac{1^8}{e^{y}+e^{-y}+1}+\frac{2^8}{e^{2y}+e^{-2y}+1}+\&c=\frac{x}{27}(1+8x)Z^9$$

$$1+2e^{-\frac{\pi y}{\sin \pi h}}+2e^{-\frac{4\pi y}{\sin \pi h}}+\&c$$

$$=\sqrt{1+\frac{h(1-h)}{(1!)^2}x+\&c}$$

where $y=\dfrac{1+\frac{h(1-h)}{(1!)^2}(1-x)+\&c}{1+\frac{h(1-h)}{(1!)^2}x+\&c}$

& u can be expressed in radicals in terms of x & h.

$$\phi=\theta+3\left\{\frac{\sin 2\theta}{1+2\cosh y}+\frac{\sin 4\theta}{2(1+2\cosh 2y)}+\&c\right\}$$

$$\theta Z=\int_0^{\phi}\left\{1+\frac{1\cdot 2}{3^2}\cdot\frac{2}{1}\cdot x\sin^2\phi+\frac{1\cdot 2\cdot 4\cdot 5}{3^2\cdot 6^2}\cdot\frac{2\cdot 4}{1\cdot 3}x\sin^4\phi+\&c\right\}d\phi$$

$$=\int_0^{\phi}\frac{\cos\left\{\frac{1}{3}\sin^{-1}(\sqrt{x}\sin\phi)\right\}}{\sqrt{1-x\sin^2\phi}}d\phi$$

But we know $\frac{x}{e^x+1} + \frac{x}{e^{2x}+1} + \frac{x}{e^{3x}+1} + \&c$

$= \left\{ \frac{x}{e^x-1} + \frac{x}{e^{2x}-1} + \frac{x}{e^{3x}-1} + \&c \right\} - \left\{ \frac{2x}{e^{2x}-1} + \frac{2x}{e^{4x}} + \frac{2x}{e^{6x}-1} + \&c \right\}$

$\therefore \psi(x) - \psi(2x) = \log_e 2$. Hence $\psi(x) = -\log x$

Ex. 1. Shew that the constant in the Series

$\sqrt[100]{1} + \sqrt[100]{2} + \sqrt[100]{3} + \sqrt[100]{4} + \cdots\cdots + \sqrt[100]{x}$ is $-\underline{.4909100}$

2. Shew that $\frac{1}{2+1} + \frac{1}{2^2+1} + \frac{1}{2^3+1} + \frac{1}{2^4+1} + \&c$

$= \frac{3}{4} + \frac{1}{48}\log_e 2$ very nearly.

3. $\frac{1}{1+1\frac{1}{9}} + \frac{1}{1+(1\frac{1}{9})^2} + \frac{1}{1+(1\frac{1}{9})^3} + \frac{1}{1+(1\frac{1}{9})^4} + \&c$

$= 6.331009$

4. $\frac{1}{1\frac{1}{9}-1} + \frac{1}{(1\frac{1}{9})^2-1} + \frac{1}{(1\frac{1}{9})^3-1} + \frac{1}{(1\frac{1}{9})^4-1} + \&c$

$= 27$ nearly

7. i. $\frac{1}{x-1} + \frac{1}{x^2-1} + \frac{1}{x^3-1} + \frac{1}{x^4-1} + \frac{1}{x^5-1} + \&c$

$= \frac{1}{x}\cdot\frac{x+1}{x-1} + \frac{1}{x^4}\cdot\frac{x^2+1}{x^2-1} + \frac{1}{x^9}\cdot\frac{x^3+1}{x^3-1} + \frac{1}{x^{16}}\cdot\frac{x^4+1}{x^4-1} + \&c$

ii. $\frac{1}{x-1} - \frac{1}{x^2-1} + \frac{1}{x^3-1} - \frac{1}{x^4-1} + \&c$ or

$\frac{1}{x+1} + \frac{1}{x^2+1} + \frac{1}{x^3+1} + \frac{1}{x^4+1} + \&c$

$= \frac{1}{x}\cdot\frac{x^2+1}{x^2-1} - \frac{1}{x^4}\cdot\frac{x^4+1}{x^4-1} + \frac{1}{x^9}\cdot\frac{x^6+1}{x^6-1} - \frac{1}{x^{16}}\cdot\frac{x^8+1}{x^8-1} + \&c$

Sol. $\frac{1}{x-1} = \frac{1}{x-1}$

$\frac{1}{x^2-1} = \frac{1}{x^2} + \frac{1}{x^2(x^2-1)}$

$\frac{1}{x^3-1} = \frac{1}{x^3} + \frac{1}{x^6} + \frac{1}{x^6(x^3-1)}$

$\frac{1}{x^4-1} = \frac{1}{x^4} + \frac{1}{x^8} + \frac{1}{x^{12}} + \frac{1}{x^{12}(x^4-1)}$

&c &c &c &c &c &c &c

$\frac{1}{x-1} + \frac{1}{x^2-1} + \frac{1}{x^3-1} + \frac{1}{x^4-1} + \&c =$

$$\phi^2(x) + \phi^2(x^{25}) = \{\phi(x^5) + 2x f(x^{15}, x^{35})\}^2$$

$$+ \{\phi(x^5) + 2x^4 f(x^5, x^{45})\}^2.$$

~~$\phi^2(x) - \phi^2(x^5) = 4x f(-x^5) f(-x^{20})$~~

$$\{\phi(x^{25}) + 2x f(x^{15}, x^{35})\}^2$$

$$+ \{\phi(x^{25}) + 2x^4 f(x^5, x^{45})\}^2$$

$$= \phi^2(x) - 2\phi^2(x^5) + 3\phi^2(x^{25}).$$

$$\phi^2(x) - \phi^2(x^5) = 4x\,\chi(x) f(-x^5) f(-x^{20})$$

$$\frac{1}{x}\cdot\frac{x+1}{x-1}+\frac{1}{x^4}\cdot\frac{x^2+1}{x^2-1}+\frac{1}{x^9}\cdot\frac{x^3+1}{x^3-1}+\frac{1}{x^{16}}\cdot\frac{x^4+1}{x^4-1}+\&c$$

and $\frac{1}{x-1}-\frac{1}{x^2-1}+\frac{1}{x^3-1}-\frac{1}{x^4-1}+\&c=$

$$\frac{1}{x}\cdot\frac{x^2+1}{x^2-1}-\frac{1}{x^4}\cdot\frac{x^4+1}{x^4-1}+\frac{1}{x^9}\cdot\frac{x^6+1}{x^6-1}-\frac{1}{x^{16}}\cdot\frac{x^8+1}{x^8-1}+\&c$$

8. $\frac{r}{1-ax}+\frac{r^2}{1-ax^2}+\frac{r^3}{1-ax^3}+\frac{r^4}{1-ax^4}+\&c$ to n terms

$$=\left\{\frac{arx}{1-ax}+\frac{(arx^2)^2}{1-ax^2}+\frac{(arx^3)^3}{1-ax^3}+\frac{(arx^4)^4}{1-ax^4}+\&c \text{ to } n \text{ terms}\right\}$$

$$+\left\{\frac{r-r^{n+1}}{1-r}+a\,\frac{(rx)^2-(rx)^{n+1}}{1-rx}+a^2\frac{(rx^2)^3-(rx^2)^{n+1}}{1-rx^2}+\&c \text{ to } n \text{ terms}\right\}$$

Sol. $\frac{r}{1-ax}=\frac{arx}{1-ax}+r$

$$\frac{r^2}{1-ax^2}=\frac{(arx^2)^2}{1-ax^2}+r^2+ar^2x^2$$

$$\frac{r^3}{1-ax^3}=\frac{(arx^3)^3}{1-ax^3}+r^3+ar^3x^3+a^2r^3x^6$$

$$\frac{r^4}{1-ax^4}=\frac{(arx^4)^4}{1-ax^4}+r^4+ar^4x^4+a^2r^4x^8+a^3r^4x^{12}$$

&c &c &c &c &c &c &c

Adding up all these to n terms we can get the result.

Cor. $\frac{r}{1-ax}+\frac{r^2}{1-ax^2}+\frac{r^3}{1-ax^3}+\frac{r^4}{1-ax^4}+\&c$

$$=\left\{\frac{arx}{1-ax}+\frac{(arx^2)^2}{1-ax^2}+\frac{(arx^3)^3}{1-ax^3}+\frac{(arx^4)^4}{1-ax^4}+\&c\right\}$$

$$+\left\{\frac{r}{1-r}+\frac{a(rx)^2}{1-rx}+\frac{a^2(rx^2)^3}{1-rx^2}+\frac{a^3(rx^3)^4}{1-rx^3}+\&c\right\}$$

Sol. Make n infinite in the above theorem.

Ex. $\left(\frac{1}{x-1}-\frac{1}{x}\right)+\left(\frac{1}{x^2-1}-\frac{1}{x^4}\right)+\left(\frac{1}{x^3-1}-\frac{1}{x^9}\right)+\&c$

$$=\frac{2}{x^{1\cdot2}-x^{1^2}}+\frac{2}{x^{2\cdot3}-x^{2^2}}+\frac{2}{x^{3\cdot4}-x^{3^2}}+\frac{2}{x^{4\cdot5}-x^{4^2}}+\&c$$

$$2^n + 6^n + 12^n + 20^n + \&c$$

$$= A_n + \frac{n}{\lfloor 1} A_{n+1} + \frac{n(n-1)}{\lfloor 2} A_{n+2} + \&c$$

where $A_n = (1^n + 2^n + 3^n + \&c)(1 + \cos \pi n)$

$$\log 2 \left\{ \frac{1}{2 \log 2} - \frac{1}{3 \log 3} + \frac{1}{4 \log 4} - \&c \right\}$$

$$+ (\log 2)^2 \left\{ \frac{1}{2 \log 2 \log 4} + \frac{1}{3 \log 3 \log 6} + \frac{1}{4 \log 4 \log 8} + \&c \right\} = 1$$

$$\int_0^\infty \frac{\cos 2nx}{\cosh \pi\sqrt{x} + \cos \pi\sqrt{x}} \, dx$$

$$= \frac{e^{-n}}{\cosh \frac{\pi}{2}} - \frac{3e^{-9n}}{\cosh \frac{3\pi}{2}} + \frac{5e^{-25n}}{\cosh \frac{5\pi}{2}} - \&c$$

If $\alpha\beta = \pi^3/4$, then

$$\frac{1}{\cosh\sqrt{\alpha} + \cos\sqrt{\alpha}} - \frac{1}{3} \cdot \frac{1}{\cosh\sqrt{3\alpha} + \cos\sqrt{3\alpha}} +$$

$$+ \frac{1}{\cosh \frac{\pi}{2} \cosh \beta} - \frac{1}{3} \cdot \frac{1}{\cosh \frac{3\pi}{2} \cosh 9\beta} + \&c$$

$$= \frac{\pi}{8}.$$

1. $1^n + 2^n + 3^n + 4^n + 5^n + \cdots + x^n = \phi_n(x)$

$$\phi_n(x) = \frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2} + \frac{x^{n+1}}{n+1} + \frac{x^n}{2} + B_2 \frac{n}{\lfloor 2} x^{n-1} - B_4 \frac{n(n-1)(n-2)}{\lfloor 4} x^{n-3}$$
$$+ B_6 \frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 6} x^{n-5} + \&c.$$

Sol. The corrected series is found by applying VI 1.
Let C_n be the constant. Since $\phi_n(0) = 0$ $\phi_n(x)$ must be divisible by x. The coeff. of $x^{n-n} = -\frac{\lfloor n}{\lfloor n-n \lfloor n+1} B_{n+1} \cos\frac{\pi(n+1)}{2}$

∴ The term independent of $x = -\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$.

∴ $C_n = \frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$.

2. $(a+b)^n + (a+2b)^n + (a+3b)^n + \cdots + (a+xb)^n = b^n\left\{\phi_n\left(x+\frac{a}{b}\right) - \phi_n\left(\frac{a}{b}\right)\right\}$

3. $1^n - 2^n + 3^n - 4^n + \&c = (2^{n+1}-1)\frac{B_{n+1}}{n+1}\sin\frac{\pi n}{2}$

Sol. $(1-2^{n+1})C_n = (1-2^{n+1})(1^n+2^n+3^n+\&c) = (1^n+2^n+3^n+\&c)$
$- 2^{n+1}(1^n+2^n+3^n+4^n+\&c) = 1^n - 2^n + 3^n - 4^n + \&c$

Cor. $\phi_n(-\tfrac{1}{2}) = \left(2 - \frac{1}{2^n}\right)\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$

Sol. $\phi_n(-\tfrac{1}{2}) = 1^n - (\tfrac{1}{2})^n + 2^n - (1\tfrac{1}{2})^n + 3^n - (2\tfrac{1}{2})^n + \&c$
$= -\frac{1}{2^n}(1^n - 2^n + 3^n - \&c) = \left(2 - \frac{1}{2^n}\right)\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$.

4. $\frac{B_{1-n}}{1-n}\sin\frac{\pi n}{2} = S_n = \frac{(2\pi)^n}{2\lfloor n} B_n$

From this we can find B_n for negative values of n.

Sol. $\frac{B_{1+n}}{1+n}\cos\frac{\pi(1+n)}{2}$ is the constant of $1^n + 2^n + 3^n + \&c$

∴ $\frac{B_{1-n}}{1-n}\cos\frac{\pi(1-n)}{2}$ is that of $\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c = S_n$

∴ $\frac{B_{1-n}}{1-n}\sin\frac{\pi n}{2} = \frac{(2\pi)^n}{2\lfloor n} B_n$

Since $C_{-n} = S_n$ S_{-n} is invariably written for C_n.

Ex. 1. $B_{-2} = 2S_3$; $B_{-4} = -4S_5$; $B_{-6} = 6S_7$; $B_{-8} = -8S_9$ &c

2. $\lfloor -\tfrac{1}{2} = \sqrt{\pi}$. Sol. $-\frac{B_{1\frac{1}{2}}}{1\frac{1}{2}}\sin\frac{\pi}{4} = \frac{(4\pi)^{-\frac{1}{2}}}{2\lfloor -\frac{1}{2}} B_{-\frac{1}{2}}$

If $f(x) = \Sigma \frac{p_n}{p_n - x}$ & $\phi(x) = \Sigma \frac{Q_n}{q_n - x}$

then $f(x)\phi(y) = \Sigma \left\{ \frac{p_n}{p_n - x} \phi\left(p_n \frac{y}{x}\right) + \frac{Q_n}{q_n - y} f\left(q_n \frac{x}{y}\right) \right\}$

If $\Sigma(a_n + a_{-n}) = a_0 + (a_1 + a_{-1}) + (a_2 + a_{-2}) +$ &c

then $\Sigma(a_n + a_{-n}) \Sigma(b_n + b_{-n})$

$= \Sigma(a_n b_n + a_{-n} b_{-n}) + \{ \Sigma(a_{n+1} b_{n-1} + a_{1-n} b_{-1-n})$

$+ \Sigma(a_{-n-1} b_{-n+1} + a_{n-1} b_{n+1}) \}$

$+ \{ \Sigma(a_{n+2} b_{n-2} + a_{2-n} b_{-n-2}) + \Sigma(a_{-n-2} b_{-n+2} + a_{n-2} b_{n+2}) \}$

$+$ &c

$+ \{ \Sigma(a_{1-n} b_n + a_{1+n} b_{-n}) + \Sigma(a_{-n} b_{n-1} + a_n b_{-n-1}) \}$

$+ \{ \Sigma(a_{2-n} b_{1+n} + a_{2+n} b_{1-n}) + \Sigma(a_{-1-n} b_{n-2} + a_{-1+n} b_{-n-2}) \}$

$+$ &c

$$\pi^2 x^2 \frac{\cos\theta x \cosh\theta x}{\sin\pi x \sinh\pi x}$$

$$= 1 + 4\pi x^4 \left\{ \frac{\cos\theta \cosh\theta}{(1^4 - x^4)\sinh\pi} - \frac{2\cos 2\theta \cosh 2\theta}{(2^4 - x^4)\sinh 2\pi} + \text{\&c} \right.$$

Again $\frac{B_{-\frac{1}{2}}}{-\frac{1}{2}} \sin\frac{\pi}{4} = \frac{(2\pi)^{1\frac{1}{2}}}{2\sqrt{2}} B_{1\frac{1}{2}}$.

Multiplying the two results we have

$\frac{1}{2}\cdot\frac{4}{3} B_{-\frac{1}{2}} B_{1\frac{1}{2}} = \frac{2\pi}{2\sqrt{2}} \cdot \frac{1}{\lfloor -\frac{1}{2}} B_{1\frac{1}{2}} B_{-\frac{1}{2}}$ or $\frac{2}{3} = \frac{2\pi}{3\lfloor \frac{1}{2} \lfloor -\frac{1}{2}}$

∴ $\lfloor \frac{1}{2} \lfloor -\frac{1}{2} = \pi$ ∴ $\lfloor -\frac{1}{2} = \sqrt{\pi}$.

3. In a similar manner we can prove that $\lfloor n-1 \lfloor -n = \frac{\pi}{\sin \pi n}$

4. Shew that

$\pi\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{6}} - \frac{1}{\sqrt{6}+\sqrt{8}} + \&c\right)$

$= \frac{1}{1\sqrt{1}} + \frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} + \frac{1}{7\sqrt{7}} + \&c$

Sol. L.H.S $= \frac{\pi}{\sqrt{2}}\left\{1 - (\sqrt{2}-1) + (\sqrt{3}-\sqrt{2}) - (\sqrt{4}-\sqrt{3}) + \&c\right\}$

$= \pi\sqrt{2}\,(\sqrt{1} - \sqrt{2} + \sqrt{3} - \sqrt{4} + \sqrt{5} - \&c)$ by VIII 5 ex l.

$= 2(2\sqrt{2}-1)\,\frac{B_{1\frac{1}{2}}}{\lfloor \frac{1}{2}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{\pi}{\sqrt{2}}$ by IX 3.

$= \frac{8}{3}\,\frac{\pi}{\sqrt{2}}\left(1 - \frac{1}{2\sqrt{2}}\right) \cdot \frac{2\lfloor 1\frac{1}{2}}{(2\pi)^{1\frac{1}{2}}}\left(\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \&c\right)$

$= \left(1 - \frac{1}{2\sqrt{2}}\right)\left(\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \&c\right) = \frac{1}{1\sqrt{1}} + \frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} + \&c.$

5. ~~$2\pi\left(\frac{1}{\sqrt[3]{1}} - \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[3]{4}} + \&c\right)$~~ $\sqrt{3}\,\lfloor\frac{1}{3}$

$= \left(\sqrt[3]{4\pi} + \sqrt[3]{2\pi}\right)\left(\frac{1}{\sqrt[3]{1}} - \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{7}} - \frac{1}{\sqrt[3]{10}} + \&c\right)$

6. Shew that, when x becomes infinite

i. $\sqrt{2(2x+1)} - \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{x}}\right)$

$= (\sqrt{2}+1)\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \&c\right)$.

ii. $\frac{2}{3}\sqrt{(x+\frac{1}{4})(x+\frac{1}{2})(x+\frac{3}{4})} - (\sqrt{1} + \sqrt{2} + \sqrt{3} + \dots + \sqrt{x})$

$= \frac{1}{4\pi}\left(\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \&c\right)$.

iii. $\frac{2}{5}\sqrt{x(x+\frac{1}{4})(x+\frac{1}{2})(x+\frac{3}{4})(x+1)} + \frac{5}{768}(x+\frac{1}{2})$

$- (1\sqrt{1} + 2\sqrt{2} + 3\sqrt{3} + \dots + x\sqrt{x})$

$= \frac{3}{16\pi^2}\left(\frac{1}{1\sqrt{1}} + \frac{1}{4\sqrt{2}} + \frac{1}{9\sqrt{3}} + \frac{1}{16\sqrt{4}} + \&c\right)$

$$\frac{\pi}{4}\sec\frac{\pi x}{2}\,\mathrm{Sech}\,\frac{\pi y}{2}$$

$$= \frac{1\,\mathrm{Sech}\,\frac{\pi x}{2y}}{1^2+y^2} - \frac{3\,\mathrm{Sech}\,\frac{3\pi x}{2y}}{3^2+y^2} + \frac{5\,\mathrm{Sech}\,\frac{5\pi x}{2y}}{5^2+y^2} - \&c$$

$$+ \frac{1\,\mathrm{Sech}\,\frac{\pi y}{2x}}{1^2-x^2} - \frac{3\,\mathrm{Sech}\,\frac{3\pi y}{2x}}{3^2-x^2} + \frac{5\,\mathrm{Sech}\,\frac{5\pi y}{2x}}{5^2-x^2} - \&c$$

$$\frac{\pi}{4}\sec\frac{\pi x}{2}\,\mathrm{Sech}\,\frac{\pi y}{2}.\overline{\phi(xy)-\phi(-xy)}.$$

~~$$= \Sigma\left\{\frac{\mathrm{Sech}\,\frac{\pi y}{2x}}{1^2-x^2}.\overline{\phi(y)-\phi(-y)}\right.$$~~

$$= x\left\{\frac{\phi(y)-\phi(-y)}{1^2-x^2}\,\mathrm{Sech}\,\frac{\pi y}{2x} - \frac{\phi(3y)-\phi(-3y)}{3^2-x^2}\,\mathrm{Sech}\,\frac{3\pi y}{2x}\right.$$

$$\left. + \frac{\phi(5y)-\phi(-5y)}{5^2-x^2}\,\mathrm{Sech}\,\frac{5\pi y}{2x} - \&c\right\}$$

$$+ \frac{y}{i}\left\{\frac{\phi(xi)-\phi(-xi)}{1^2+y^2}\,\mathrm{Sech}\,\frac{\pi x}{2y} - \frac{\phi(3xi)-\phi(-3xi)}{3^2+y^2}\,\mathrm{Sech}\,\frac{3\pi x}{2y}\right.$$

$$\left. + \frac{\phi(5xi)-\phi(-5xi)}{5^2+y^2}\,\mathrm{Sech}\,\frac{5\pi x}{2y} - \&c\right\}$$

5. The following theorem is applied in the solution of IX 1. More general theorems will be found in

If $x=0$, the value of the generating function of the Series

$$x^n\phi(0)+\frac{n}{\lfloor 1}x^{n-1}\phi(1)+\frac{n(n-1)}{\lfloor 2}x^{n-2}\phi(2)+\&c = \phi(n).$$

Sol. The given Series =

$$\phi(0)\left\{x^n+\frac{n}{\lfloor 1}x^{n-1}+\frac{n(n-1)}{\lfloor 2}x^{n-2}+\&c\right\}$$

$$+\phi'(0)\left\{\frac{1\,n}{\lfloor 1}x^{n-1}+\frac{2\,n(n-1)}{\lfloor 2}x^{n-2}+\frac{3\,n(n-1)(n-2)}{\lfloor 3}x^{n-3}+\&c\right\}$$

$$+\frac{1}{2}\phi''(0)\left\{\frac{1^2 n}{\lfloor 1}x^{n-1}+\frac{2^2 n(n-1)}{\lfloor 2}x^{n-2}+\frac{3^2 n(n-1)(n-2)}{\lfloor 3}x^{n-3}+\&c\right\}$$

$$+\ \&c.$$

$$=\phi(0)(1+x)^n+n\phi'(0)(1+x)^{n-1}+\frac{1}{2}\phi''(0)\left\{n(n-1)(1+x)^{n-2}+n(1+x)^{n-1}\right\}$$

$$+\&c = \phi(0)+\frac{n}{\lfloor 1}\phi'(0)+\frac{n^2}{\lfloor 2}\phi''(0)+\&c \text{ when } x=0$$

$$=\phi(n).$$

Cor. When $x=0$

$$\frac{\phi(1)}{x}-\frac{\phi(2)}{x^2}+\frac{\phi(3)}{x^3}-\frac{\phi(4)}{x^4}+\&c=\phi(0).$$

Sol. Write -1 for ~~x~~ in the above theorem.

Eg. ~~Let~~ $\phi(n)=\frac{1}{n}\sin\frac{\pi n}{2}$ then $\phi(0)=\frac{\pi}{2}$.

$\therefore$ When $x=0$ $\quad \frac{1}{x}-\frac{1}{3x^3}+\frac{1}{5x^5}-\frac{1}{7x^7}+\&c=\frac{\pi}{2}$

which is same as saying $\tan^{-1}\infty=\frac{\pi}{2}$.

Ex. 1. If $x=0$ Show that

$$\frac{\pi}{2\sqrt{x}}-\frac{1}{2}\cdot\frac{1}{x\sqrt{x}}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{x^3\sqrt{x}}-\frac{1.3.5.7.9}{2.4.6.8.10}\cdot\frac{1}{x^5\sqrt{x}}+\&c=\sqrt{2}$$

2. If $x=0$ then $\frac{1}{x}-\frac{\lfloor 1}{x^2}+\frac{\lfloor 2}{x^3}-\frac{\lfloor 3}{x^4}+\&c=\infty$. $\quad = \frac{1}{1}-\frac{1^2}{3}+\frac{2^2}{5}+\frac{3^2}{7}-\&c$ when $x=0$

N.B. Here $L.H.S = \cfrac{1}{x+1-\cfrac{1^2}{x+3-\cfrac{2^2}{x+5-\cfrac{3^2}{x+7-\&c}}}}$

$$=1+\frac{1}{2}+\frac{1}{3}+\&c=\infty.$$

or $L.H.S=\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}(1+\frac{1}{2})+\frac{x^3}{\lfloor 3}(1+\frac{1}{2}+\frac{1}{3})+\&c - e^x(C_0+\log_e x)$

$=\infty$ when $x=0$.

3. If $x=0$ then $x^n+\frac{n}{\lfloor 1}x^{n-1}+n(n-1)x^{n-2}+n(n-1)(n-2)x^{n-3}+\&c$

$$=\lfloor n \text{ for all values of } n$$

$$\pi^2 x^2 \cot \pi x \coth \pi x$$
$$= 1 - 4\pi x^4 \left\{ \frac{\coth \pi}{1^4 - x^4} + \frac{2 \coth 2\pi}{2^4 - x^4} + \frac{3 \coth 3\pi}{3^4 - x^4} + \&c \right\}.$$

$$\pi^2 xy \cot \pi x \coth \pi y$$
$$= 1 + 2\pi xy \left\{ \frac{\coth \frac{\pi x}{y}}{1 + y^2} + \frac{2 \coth \frac{2\pi x}{y}}{2^2 + y^2} + \&c \right\}$$
$$- 2\pi xy \left\{ \frac{\coth \frac{\pi y}{x}}{1^2 - x^2} + \frac{2 \coth \frac{2\pi y}{x}}{2^2 - x^2} + \&c \right\}$$

$$(\pi x)^2 \cdot \frac{\cosh \pi x\sqrt{2} + \cos \pi x \sqrt{2}}{\cosh \pi x \sqrt{2} - \cos \pi x \sqrt{2}}.$$
$$= 1 + 4\pi x^4 \left\{ \frac{\coth \pi}{1^4 + x^4} + \frac{2 \coth 2\pi}{2^4 + x^4} + \frac{3 \coth 3\pi}{3^4 + x^4} + \&c \right\}$$

$$\frac{\pi}{8} \sec \frac{\pi x}{2} \operatorname{sech} \frac{\pi x}{2}$$
$$= \frac{1^3 \operatorname{sech} \frac{\pi}{2}}{1^4 - x^4} - \frac{3^3 \operatorname{sech} \frac{3\pi}{2}}{3^4 - x^4} + \frac{5^3 \operatorname{sech} \frac{5\pi}{2}}{5^4 - x^4} - \&c$$

$$\frac{\pi}{4} \cdot \frac{1}{\cosh \frac{\pi x}{\sqrt{2}} + \cos \frac{\pi x}{\sqrt{2}}}$$
$$= \frac{1^3 \operatorname{sech} \frac{\pi}{2}}{1^4 + x^4} - \frac{3^3 \operatorname{sech} \frac{3\pi}{2}}{3^4 + x^4} + \frac{5^3 \operatorname{sech} \frac{5\pi}{2}}{5^4 + x^4} - \&c$$

$$\int_0^\infty \frac{\cos nx}{4 + x^4} dx = \frac{\pi}{8} e^{-n} (\cos n + \sin n).$$

4. If $x=0$ Shew that

$$\frac{1}{x} - \frac{\lfloor 2}{x^3} + \frac{\lfloor 4}{x^5} - \frac{\lfloor 6}{x^7} + \&c = \frac{\pi}{2}.$$

Sol. Write $\lfloor n-1 \sin\frac{\pi n}{2}$ for $\phi(n)$ in the above cor.

Then $\phi(0) = \lfloor n \cdot \frac{\sin\frac{\pi n}{2}}{n}$ when $n=0 = \frac{\pi}{2}$.

N.B. Thus we are able to find exact values when $x=0$ from the above theorem and cor. though the generating functions may be too difficult to find.

The generating function in ex. 4.

$$= \frac{\pi}{2}\cos x + (C_0 + \log_e x)\sin x - \left\{1\frac{x}{\lfloor 1} - \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^3}{\lfloor 3} + \&c\right\}$$

$= \frac{\pi}{2}$ when $x=0$.

6. $(a+b)^n - (a+2b)^n + (a+3b)^n - \&c = b^n\left\{\phi_n\left(\frac{a}{2b}\right) - \phi_n\left(\frac{a-b}{2b}\right)\right\}$

Ex. (a) If $x^2+x=y$ and $x+\frac{1}{2}=a$ shew that

(1) $\phi_1(x) = \frac{y}{2}$; (2) $\phi_2(x) = a\frac{y}{3}$; (3) $\phi_3(x) = \frac{y^2}{4}$; (4) $\phi_4(x) = \frac{a}{5}y\left(y-\frac{1}{3}\right)$;

(5) $\phi_5(x) = \frac{y^2}{6}\left(y-\frac{1}{2}\right)$; (6) $\phi_6(x) = \frac{a}{7}y\left(y^2-y+\frac{1}{3}\right)$; (7) $\phi_7(x) = \frac{y^2}{8}\left(y^2-\frac{4}{3}y+\frac{2}{3}\right)$

(8) $\phi_8(x) = \frac{a}{9}y\left(y^3-2y^2+\frac{9}{5}y-\frac{3}{5}\right)$ (9) $\phi_9(x) = \frac{y^2}{10}\left(y^3-\frac{5}{2}y^2+3y-1\frac{1}{2}\right)$

(10) $\phi_9(x) = \frac{y^2}{10}(y-1)\left(y^2-\frac{3}{2}y+\frac{3}{2}\right)$; (11) $\phi_{10}(x) = \frac{a}{11}y(y-1)\left(y^3-\frac{7}{3}y^2+\frac{10}{3}y-\frac{5}{3}\right)$

(12) $\phi_{11}(x) = \frac{y^2}{12}\left(y^4-4y^3+8\frac{1}{2}y^2-10y+5\right)$.

(b). 1. $y^2 = 4\phi_3(x) = 4\{\phi_1(x)\}^2$

2. $ay^2 = 5\phi_4(x) + \phi_2(x) = 6\phi_1(x)\phi_2(x)$

3. $y^3 = 6\phi_5(x) + 2\phi_3(x) = 8\phi_1(x)\phi_3(x)$

4. $ay^3 = 7\phi_6(x) + 5\phi_4(x) = 12\phi_2(x)\phi_3(x)$.

5. $y^4 = 8\phi_7(x) + 8\phi_5(x) = 16\phi_3(x)\phi_3(x)$.

6. $ay^4 = 9\phi_8(x) + 14\phi_6(x) + \phi_4(x) = 24\phi_1(x)\phi_2(x)\phi_3(x)$.

7. $y^5 = 10\phi_9(x) + 20\phi_7(x) + 2\phi_5(x)$.

8. $ay^5 = 11\phi_{10}(x) + 30\phi_8(x) + 7\phi_6(x)$.

9. $y^6 = 12\phi_{11}(x) + 40\phi_9(x) + 12\phi_7(x)$.

7. If n is a +ve integer then $\phi_n(x-1) + (-1)^n\phi_n(-x) = 0$

$$\frac{\left|\underline{\frac{x+l+n-m-1}{2}}\right. \left|\underline{\frac{x+l-n-m-1}{2}}\right. \left|\underline{\frac{x-l+n+m-1}{2}}\right. \left|\underline{\frac{x-l-n+m-1}{2}}\right.}{\left|\underline{\frac{x-l+n-m-1}{2}}\right. \left|\underline{\frac{x-l-n-m-1}{2}}\right. \left|\underline{\frac{x+l+n+m-1}{2}}\right. \left|\underline{\frac{x+l-n+m-1}{2}}\right.} = P$$

then $$\frac{1-P}{1+P} = \frac{2lmx}{x^2+l^2+m^2-n^2-1} + \frac{4(x^2-1)(l^2-1)(m^2-1)}{3(x^2+l^2+m^2-n^2-5)} + \&c$$

The expansion only is true.

$$\frac{1}{x^2} - \frac{2\cos\theta}{1^2-x^2} - \frac{2\cos 2\theta}{2^2-x^2} - \frac{2\cos 3\theta}{3^2-x^2} - \&c$$

$$= \frac{\pi}{x}\left\{\cot \pi x \cos\theta x + \sin\theta x\right\}$$

$$\frac{\sin\theta}{1^2-x^2} - \frac{\sin 3\theta}{3^2-x^2} + \frac{\sin 5\theta}{5^2-x^2} - \&c = \frac{\pi}{4x}\sec\frac{\pi x}{2}\sin\theta x.$$

Sol. Let $\psi(x) = \phi_r(x-1) + (-1)^r \phi_r(-x)$, then we see that $\psi(x+1) - \psi(x) = 0$. ∴ If r is a positive integer $\psi(x) = 0$.

Ex. 1. If r is odd shew that $\phi_r(x-1) = \phi_r(-x)$.

2. Shew that $\phi_r(x)$ is divisible by $x^2(x+1)^2$ or $x(x+\frac{1}{2})(x+1)$ according as r is odd or even and point out the exceptional case; in both cases r being a positive integer.

8. $\phi_r(x) = -B_r x \cos\frac{\pi r}{2} - \frac{r}{\lfloor 2} B_{r-1} x^2 \sin\frac{\pi r}{2} + \frac{r(r-1)}{\lfloor 3} B_{r-2} x^3 \cos\frac{\pi r}{2}$

$+ \frac{r(r-1)(r-2)}{\lfloor 4} B_{r-3} x^4 \sin\frac{\pi r}{2} - \&c$

$= -r x S_{1-r} - \frac{r(r-1)}{\lfloor 2} x^2 S_{2-r} - \frac{r(r-1)(r-2)}{\lfloor 3} x^3 S_{3-r} - \&c.$

Sol. Apply VIII 3.

9. $\phi_r(x) = 1^r - (1+x)^r + 2^r - (2+x)^r + 3^r - (3+x)^r + \&c.$

10. $\phi_r(x) - n^r \left\{ \phi_r\left(\frac{x}{n}\right) + \phi_r\left(\frac{x-1}{n}\right) + \phi_r\left(\frac{x-2}{n}\right) + \dots + \phi_r\left(\frac{x-n+1}{n}\right) \right\}$

$= (n^{r+1} - 1) \frac{B_{r+1}}{r+1} \sin\frac{\pi r}{2} = (1 - n^{r+1}) S_{-r}$

Sol. Apply VIII 4.

Cor. $\phi_r(-\frac{1}{n}) + \phi_r(-\frac{2}{n}) + \phi_r(-\frac{3}{n}) + \dots + \phi_r(-\frac{n-1}{n}) = (n - n^{-r}) S_{-r}$

$= (n^{-r} - n) \frac{B_{r+1}}{r+1} \sin\frac{\pi r}{2}$.

11. If r is a negative integer, then

$\phi_r(x-1) + (-1)^r \phi_r(-x) = \{1 + (-1)^r\} S_r + \frac{(-1)^r d_{-(r+1)x} \pi \cot \pi x}{\lfloor -r-1}$

Sol. From X we have $\phi_{-1}(x-1) - \phi_{-1}(x) = -\pi \cot \pi x$

Differentiate both sides r times.

N.B. The following method is very useful in finding the derivatives of $\pi \cot \pi x$. Let $\cot \pi x = y$

The coeff^ts of the coeff^t of π^n are the same as those in the expansion of $(\tan^{-1}\frac{1}{y})^{-n}$

Each derivative is divisible by y^2+1 so that the last term can be exactly found.

$$\left\{ \left\lfloor\frac{\alpha+\beta+\gamma+\delta-\epsilon-1}{2}\right. \left\lfloor\frac{\alpha+\beta+\gamma-\delta+\epsilon-1}{2}\right. \left\lfloor\frac{\alpha+\beta-\gamma+\delta+\epsilon-1}{2}\right. \left\lfloor\frac{\alpha-\beta+\gamma+\delta+\epsilon-1}{2}\right. \times \right.$$

$$\left. \left\lfloor\frac{\alpha+\beta-\gamma-\delta-\epsilon-1}{2}\right. \left\lfloor\frac{\alpha-\beta+\gamma-\delta-\epsilon-1}{2}\right. \left\lfloor\frac{\alpha-\beta-\gamma+\delta-\epsilon-1}{2}\right. \left\lfloor\frac{\alpha-\beta-\gamma-\delta+\epsilon-1}{2}\right. \right\} = Q$$

$$\left\{ \left\lfloor\frac{\alpha+\beta+\gamma+\delta+\epsilon-1}{2}\right. \left\lfloor\frac{\alpha+\beta+\gamma-\delta-\epsilon-1}{2}\right. \left\lfloor\frac{\alpha+\beta-\gamma-\delta+\epsilon-1}{2}\right. \left\lfloor\frac{\alpha-\beta-\gamma+\delta+\epsilon-1}{2}\right. \times \right.$$

$$\left. \left\lfloor\frac{\alpha-\beta+\gamma+\delta-\epsilon-1}{2}\right. \left\lfloor\frac{\alpha-\beta+\gamma-\delta+\epsilon-1}{2}\right. \left\lfloor\frac{\alpha+\beta-\gamma+\delta-\epsilon-1}{2}\right. \left\lfloor\frac{\alpha-\beta-\gamma-\delta-\epsilon-1}{2}\right. \right\} = P$$

$$\neq \frac{8\alpha\beta\gamma\delta\epsilon}{\left\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\epsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2-1)^2-2^2\right\}} +$$

$$\frac{64(\alpha^2-1)(\beta^2-1)(\gamma^2-1)(\delta^2-1)(\epsilon^2-1)}{3\left\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\epsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2-5)^2-6^2\right\}} + \text{\&c}$$

$= \frac{P-Q}{P+Q}$. if any one of $\alpha, \beta, \gamma, \delta, \epsilon$ be an integer.

πy

$\pi^2(y^2+1)$

$\pi^3(y^3+y)$

$\pi^4(y^4+\frac{4}{3}y^2+\frac{1}{3})$

$\pi^5(y^5+\frac{5}{3}y^3+\frac{2}{3}y)$

$\pi^6(y^6+2y^4+\frac{17}{15}y^2+\frac{2}{15})$

$\pi^7(y^7+\frac{7}{3}y^5+\frac{77}{45}y^3+\frac{17}{45}y)$

$\pi^8(y^8+\frac{8}{3}y^6+\frac{12}{5}y^4+\frac{248}{315}y^2+\frac{17}{315})$

$\pi^9(y^9+3y^7+\frac{16}{5}y^5+\frac{88}{63}y^3+\frac{62}{315}y)$

$\pi^{10}(y^{10}+\frac{10}{3}y^8+\frac{37}{9}y^6+\frac{424}{189}y^4+\frac{1382}{2835}y^2+\frac{62}{2835})$

Write under each term the quotient obtained by dividing the sum of the products of the coeff^t and the index of that term and of the preceding term by the index of π.

Ex. (a) For all values of r shew that

1. $\phi_r(x) - 2^r\left\{\phi_r\left(\frac{x}{2}\right) + \phi_r\left(\frac{x-1}{2}\right)\right\} = (1-2^{r+1})S_{-r}$.
2. $\phi_r(-\frac{1}{2}) = (2-\frac{1}{2^r})S_{-r}$
3. $\phi_r(-\frac{1}{3}) + \phi_r(-\frac{2}{3}) = (3-\frac{1}{3^r})S_{-r}$
4. $\phi_r(-\frac{1}{4}) + \phi_r(-\frac{3}{4}) = (2+\frac{1}{2^r}-\frac{1}{4^r})S_{-r}$
5. $\phi_r(-\frac{1}{6}) + \phi_r(-\frac{5}{6}) = (1+\frac{1}{2^r}+\frac{1}{3^r}-\frac{1}{6^r})S_{-r}$.

(b) If r is a positive odd integer shew that

1. $\phi_r(-\frac{1}{3}) = (3-\frac{1}{3^r})\frac{S_{-r}}{2}$
2. $\phi_r(-\frac{1}{4}) = (1+\frac{1}{2^{r+1}}-\frac{1}{2^{2r+1}})S_{-r}$
3. $\phi_r(-\frac{1}{6}) = (1+\frac{1}{2^r}+\frac{1}{3^r}-\frac{1}{6^r})\frac{S_{-r}}{2}$
4. $\phi_r(-\frac{1}{5}) + \phi_r(-\frac{2}{5}) = (5-\frac{1}{5^r})\frac{S_{-r}}{2}$
5. $\phi_r(-\frac{1}{8}) + \phi_r(-\frac{3}{8}) = (2+\frac{1}{2^{2r+1}}-\frac{1}{2^{3r+1}})S_{-r}$
6. $\phi_r(-\frac{1}{10}) + \phi_r(-\frac{3}{10}) = (5+\frac{1}{5^r}-\frac{1}{10^r})\frac{S_{-r}}{2}$.
7. $\phi_r(-\frac{1}{12}) + \phi_r(-\frac{5}{12}) = (6+\frac{1}{6^r}-\frac{1}{12^r})\frac{S_{-r}}{2}$.

(c) Shew that IX 11 is true even for positive integral values of r and hence deduce IX 7 from IX 11.

$$\frac{\phi(\log 1)}{1} + \frac{\phi(\log 2)}{2} + \frac{\phi(\log 3)}{3} + $$

$$= \int_0^\infty \phi(x)\,dx + c_0\phi(0) + \frac{c_1}{\underline{1}}\phi'(0) + $$

$$2x + \frac{2^x - 2^{-x}}{1.2} + \frac{3^x - 3^{-x}}{2.3} + \frac{4^x - 4^{-x}}{3.4} + \&c$$

$$= \frac{1}{x} - \pi\cot\pi x + \&c$$

$$\frac{1}{2(2^x-1)} + \frac{1}{3(3^x-1)} + \frac{1}{4(4^x-1)} + $$

$$= \frac{\cdot 7946786 - \log_e x}{x} + \cdot 2113922$$

$$- \cdot 0060680x - \cdot 0000028x^3 + \&c$$

$$\frac{1}{2\log_e 2} + \frac{1}{3\log_e 3} + \frac{1}{4\log_e 4} + \cdots + \frac{1}{n\log_e n}$$

$$= \cdot 7946786 + \log_e\log_e(n + \tfrac{1}{2}) \text{ nearly.}$$

$$\frac{1}{2^{x+1}\log_e 2} + \frac{1}{3^{x+1}\log_e 3} + \frac{1}{4^{x+1}\log_e 4} + $$

$$= \sout{\cdot 7946786} - \log_e x + \cdot 4227843\sout{351}x$$

$$- \cdot 0364079\sout{2274}x^2 + \cdot 001617x^3$$

$$+ \cdot 000085x^4 - \cdot 00002x^5 - \&c$$

$$+ \cdot 2174630$$

(d). 1. $\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \&c = \frac{7}{8} S_3$.

2. $\frac{1}{1^3} + \frac{1}{4^3} + \frac{1}{7^3} + \frac{1}{10^3} + \&c = \frac{2\pi^3}{81\sqrt{3}} + \frac{13}{27} S_3$.

3. $\frac{1}{1^3} + \frac{1}{5^3} + \frac{1}{9^3} + \frac{1}{13^3} + \&c = \frac{\pi^3}{64} + \frac{7}{16} S_3$.

4. $\frac{1}{1^3} + \frac{1}{7^3} + \frac{1}{13^3} + \frac{1}{19^3} + \&c = \frac{\pi^3}{36\sqrt{3}} + \frac{91}{216} S_3$.

12. $2^n\left\{\phi_n(-\tfrac{1}{6}) - \phi_n(-\tfrac{5}{6})\right\} = (2^n+1)\left\{\phi_n(-\tfrac{1}{3}) - \phi_n(-\tfrac{2}{3})\right\}$.

Sol. $\phi_n(-\tfrac{1}{3}) - 2^n\left\{\phi_n(-\tfrac{1}{6}) + \phi_n(-\tfrac{2}{3})\right\} = (2^{n+1}-1) S_{-n}$

& $\phi_n(-\tfrac{2}{3}) - 2^n\left\{\phi_n(-\tfrac{1}{3}) + \phi_n(-\tfrac{5}{6})\right\} = (2^{n+1}-1) S_{-n}$ } by IX 10.

$\therefore 2^n\left\{\phi_n(-\tfrac{1}{6}) - \phi_n(-\tfrac{5}{6})\right\} = (2^n+1)\left\{\phi_n(-\tfrac{1}{3}) - \phi_n(-\tfrac{2}{3})\right\}$.

13. If c_n be the constant of $\frac{(\log_e 1)^n}{1} + \frac{(\log_e 2)^n}{2} + \frac{(\log_e 3)^n}{3} + \&c$ then

$S_{n+1} = \frac{1}{n} + c_0 - \frac{n}{\lfloor 1} c_1 + \frac{n^2}{\lfloor 2} c_2 - \frac{n^3}{\lfloor 3} c_3 + \&c$

Sol. It has been proved in VII 6 ex. 1 that $S_{n+1} - \frac{1}{n}$ is finite when $n = 0$. The remaining part is obtained by applying VIII 6 Case II.

The above result may be written as follows

$\frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{4^{n+1}} + \&c$

$= \frac{1}{n} + .5772156649 + .0728158455\,n$

$\quad - (.00485\,n^2 + .00034\,n^3) + \frac{\left(\frac{n}{10}\right)^4}{1 + \frac{n}{10}}\,\theta$.

where θ may be taken as equivalent to $1 + \frac{2n}{50}$.

N.B. The above result is true for all values of n.

Ex. 1. $\frac{1}{1\sqrt[10]{1}} + \frac{1}{2\sqrt[10]{2}} + \frac{1}{3\sqrt[10]{3}} + \&c = 10.58444842$

2. $\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \&c = 2.612315$.

3. $\frac{1}{1^2\sqrt{1}} + \frac{1}{2^2\sqrt{2}} + \frac{1}{3^2\sqrt{3}} + \&c = 1.341490$.

4. $B_{1\frac{1}{2}} = .4409932$. 5. $B_{\frac{1}{2}} = -1.032627$

6. $B_{\frac{1}{3}} = -.9420745$ & $B_{-\frac{1}{3}} = -1.3841347$

$$\int_0^\infty e^{-x}\left(1+\frac{x}{n}\right)^{n-h}dx$$

$$= 1 + \left(1-\frac{h}{n}\right) + \left(1-\frac{h}{n}\right)\left(1-\frac{h+1}{n}\right) + \left(1-\frac{h}{n}\right)\left(1-\frac{h+1}{n}\right)\left(1-\frac{h+2}{n}\right) + \&c$$

$$= \frac{e^n \underline{\lvert n-h}}{2n^{n-h}} + A_0 - \frac{A_1}{n} + \frac{A_2}{n^2} - \&c$$

$$A_0 = \frac{2}{3} - h. \quad A_1 = \frac{4}{135} - \frac{h^2(1-h)}{3}$$

$$A_2 = \frac{8}{2835} + \frac{2h(1-h)}{135} - \frac{h(1-h^2)(2-3h^2)}{45}. \quad \&c$$

$$1 + \frac{\phi(h.\overline{\alpha+\delta})}{\phi(h.\overline{\beta+\gamma})} + \frac{\phi(h.\overline{\alpha+\delta})\,\phi(h.\overline{\alpha+2\delta})}{\phi(h.\overline{\beta+\gamma})\,\phi(h.\overline{\beta+2\gamma})} + \&c$$

$$= \sqrt{\frac{\pi\,\phi(0)}{2(\gamma-\delta)h\,\phi'(0)}} + \frac{1}{3}\cdot\frac{\gamma+\delta}{\gamma-\delta}\left\{1 - \frac{\phi(0)}{\phi'(0)}\cdot\frac{\phi''(0)}{\phi'(0)}\right\} + \frac{\alpha-\beta}{\gamma-\delta}.$$

if h is very small.

$$1 + \frac{x^2}{2}\cdot\frac{1}{2n+1} + \frac{x^4}{2.4}\cdot\frac{1}{(2n+1)(2n+3)} + \&c$$

$$= \frac{2^{n-1}\underline{\lvert n-\tfrac{1}{2}}}{x^n\sqrt{\pi}}\left[e^x\left\{1 - \frac{n(n-1)}{2}\cdot\frac{1}{x} + \frac{(n+1)n(n-1)(n-2)}{2.4}\cdot\frac{1}{x^2} - \&c\right\} + \theta e^{-x}\cos\pi n\right]$$

where $\theta = e^{\frac{n(n-1)}{2x+1}}$ nearly.

7. $B_{-\frac{1}{2}} = -1.847228$.

8. Shew that $S_{1+n} + S_{1-n} = \dfrac{2S_0}{1 + .00839n^2 + .0001n^4 + \&c}$

14. $\dfrac{\phi_n(x-1) - \phi_n(-x)}{4\lfloor n} = -\cos\dfrac{\pi n}{2}\left\{\dfrac{\sin 2\pi x}{(2\pi)^{n+1}} + \dfrac{\sin 4\pi x}{(4\pi)^{n+1}} + \&c\right\}$

Sol. $\phi_n(x-1) - \phi_n(-x)$

$= (1-x)^n - x^n + (2-x)^n - (1+x)^n + (3-x)^n - (2+x)^n + \&c$

Then arrange the terms in ascending powers of x and substitute $\dfrac{B_n}{n}\cos\dfrac{\pi n}{2}$ for S_{1-n}. Similarly

15. $\dfrac{\phi_n(x-1) + \phi_n(-x) - 2S_{-n}}{4\lfloor n} = \sin\dfrac{\pi n}{2}\left\{\dfrac{\cos 2\pi x}{(2\pi)^{n+1}} + \dfrac{\cos 4\pi x}{(4\pi)^{n+1}} + \&c\right\}$

N.B. The above two theorems are true for all values of x when n is an integer but when n is fractional they are true only when x lies between 0 and 1.

16. $\dfrac{(2\pi q)^n}{4\lfloor n-1}\left\{\phi_{n-1}\left(\dfrac{p}{q}-1\right) - \phi_{n-1}\left(-\dfrac{p}{q}\right)\right\}$ when $\dfrac{p}{q}$ lies between 0 & 1

$= -\sin\dfrac{\pi n}{2}\left[\left\{S_n - \phi_{-n}\left(\dfrac{1}{q}-1\right)\right\}\sin\dfrac{2\pi p}{q} + \left\{S_n - \phi_{-n}\left(\dfrac{2}{q}-1\right)\right\}\sin\dfrac{4\pi p}{q}\right.$

$\left. + \left\{S_n - \phi_{-n}\left(\dfrac{3}{q}-1\right)\right\}\sin\dfrac{6\pi p}{q} + \ldots\ldots + \left\{S_n - \phi_{-n}\left(\dfrac{q-1}{q}-1\right)\right\}\sin\dfrac{(2q-2)\pi p}{q}\right]$

17. $\dfrac{(2\pi q)^n}{4\lfloor n-1}\left\{\phi_{n-1}\left(\dfrac{p}{q}-1\right) + \phi_{n-1}\left(-\dfrac{p}{q}\right) - 2S_{1-n}\left(1 - \dfrac{1}{q^n}\right)\right\}$ for the same limits

$= -\cos\dfrac{\pi n}{2}\left[\left\{S_n - \phi_{-n}\left(\dfrac{1}{q}-1\right)\right\}\cos\dfrac{2\pi p}{q} + \left\{S_n - \phi_{-n}\left(\dfrac{2}{q}-1\right)\right\}\cos\dfrac{4\pi p}{q}\right.$

$\left. + \left\{S_n - \phi_{-n}\left(\dfrac{3}{q}-1\right)\right\}\cos\dfrac{6\pi p}{q} + \ldots + \left\{S_n - \phi_{-n}\left(\dfrac{q-1}{q}-1\right)\right\}\cos\dfrac{(2q-2)\pi p}{q}\right]$

In the above two theorems p & q are integers.

In 16 & 17 the same theorems 14 & 15 are written in another form.

Ex. 1. $\phi_n\left(-\frac{1}{4}\right) - \phi_n\left(-\frac{3}{4}\right) = 2 \cdot \dfrac{E_{n+1}}{4^{n+1}}\cos\dfrac{\pi n}{2}$.

Sol. Write $\frac{1}{4}$ for x in IX 14.

$$1 + \frac{(en)}{1} + \frac{(en)^2}{2^2} + \frac{(en)^3}{3^3} + \&c$$

$$= \sqrt{2\pi n}\, e^{n - \frac{1}{24n} - \frac{1}{48n^2} - \left(\frac{1}{36} + \frac{1}{5760}\right)\frac{1}{n^3}}$$

$$\int_0^\infty \frac{x^{n-1}\,dx}{1 + \left(\frac{x}{1}\right) + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{3}\right)^3 + \left(\frac{x}{4}\right)^4 + \&c}$$

$$= n^n \left\{ \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \frac{3}{8n^4} + \&c \right\}$$

2. $1^n - 3^n + 5^n - 7^n + \&c = \frac{1}{2} E_{n+1} \cos \frac{\pi n}{2}$.

3. $E_{1-n} \cos \frac{\pi n}{2} = \left(\frac{\pi}{2}\right)^n \frac{E_n}{\lfloor n-1}$.

4. $\pi \left\{ \frac{1}{2} - \frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} - \frac{1}{\sqrt{5}+\sqrt{7}} + \&c \right\}$

$= \frac{1}{1\sqrt{1}} - \frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} - \frac{1}{7\sqrt{7}} + \&c.$

5. $\frac{\pi}{2} \left\{ \frac{1}{\sqrt{2}} + \frac{1}{2} - \frac{1}{\sqrt{1}+\sqrt{3}} - \frac{1}{\sqrt{2}+\sqrt{4}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{4}+\sqrt{6}} - \&c \right\}$

$= \frac{1}{1\sqrt{1}} + \frac{1}{5\sqrt{5}} + \frac{1}{9\sqrt{9}} + \frac{1}{13\sqrt{13}} + \&c$

6. If $\frac{p}{q}$ lies between 0 & 1 p being any integer & q an odd integer show that

i. $\frac{(2\pi q)^n}{4\lfloor n-1} \left\{ \phi_{n-1}\left(\frac{p}{q}-1\right) - \phi_{n-1}\left(\frac{p}{q}\right) \right\}$

$= \sin \frac{\pi n}{2} \left[\left\{ \phi_{-n}\left(\frac{1}{q}-1\right) - \phi_{-n}\left(-\frac{1}{q}\right) \right\} \sin \frac{2\pi p}{q} \right.$

$\left. + \left\{ \phi_{-n}\left(\frac{2}{q}-1\right) - \phi_{-n}\left(-\frac{2}{q}\right) \right\} \sin \frac{4\pi p}{q} + \&c \text{ to } \frac{q-1}{2} \text{ terms} \right]$

ii. $\frac{(2\pi q)^n}{4\lfloor n-1} \left\{ \phi_{n-1}\left(\frac{p}{q}-1\right) + \phi_{n-1}\left(-\frac{p}{q}\right) - 2S_{1-n}\left(1-\frac{1}{q^n}\right) \right\}$

$= \cos \frac{\pi n}{2} \left[\left\{ \phi_{-n}\left(\frac{1}{q}-1\right) - \phi_{-n}\left(-\frac{1}{q}\right) \right\} \cos \frac{2\pi p}{q} + \left\{ \phi_{-n}\left(\frac{2}{q}-1\right) - \phi_{-n}\left(-\frac{2}{q}\right) \right\} \cos \frac{4\pi p}{q} \right.$

$\left. + \&c \text{ to } \frac{q-1}{2} \text{ terms} \right]$

Hence show that

7. $\frac{(6\pi)^n}{2\lfloor n-1 \sqrt{3}} \left\{ \phi_{n-1}\left(-\frac{1}{3}\right) - \phi_{n-1}\left(-\frac{2}{3}\right) \right\} = \left\{ \phi_{-n}\left(-\frac{1}{3}\right) - \phi_{-n}\left(-\frac{2}{3}\right) \right\} \sin \frac{\pi n}{2}$.

8. $\frac{2^{n-1}}{\lfloor n} \phi_n(-x) = \frac{\sin \pi x}{\pi^{n+1}} \cos\left(\pi x + \frac{\pi n}{2}\right) + \frac{\sin 2\pi x}{(2\pi)^{n+1}} \cos\left(2\pi x + \frac{\pi n}{2}\right) + \&c$

Sol. Combine the results of IX 14 & 15.

9. $\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{1-x}} + \frac{1}{\sqrt{1+x}} - \frac{1}{\sqrt{2-x}} + \frac{1}{\sqrt{2+x}} - \&c$

$= 2\left(\frac{\sin 2\pi x}{\sqrt{1}} + \frac{\sin 4\pi x}{\sqrt{2}} + \frac{\sin 6\pi x}{\sqrt{3}} + \&c \right)$

$$(p-q-1)\int_0^\infty \frac{\left(1+\frac{x}{n}\right)^q}{\left(1+\frac{x}{m}\right)^p}\,dx$$

$$= \frac{1}{2}\cdot\frac{m^p}{n^q}\cdot\frac{\lfloor q}{\lfloor p-1}\cdot\frac{\lfloor p-q-1}{(m-n)^{p-q-1}}$$

$$+ \frac{2}{3}(m+n) + mq - np$$

CHAPTER X

1. $\frac{B_r}{r}\cos\frac{\pi r}{2}+\frac{1}{r}$ when r is 0 is a finite quantity which is invariably denoted by the symbol C_0 the value of which can be found from IX 2. It is the constant of S_1 & its value is $.5772156649015328606\ldots$ & $e^{-C_0} = .56|45,948356$

Sol. Since L.H.S in IX 1 is finite when $r=-1$

$\frac{B_r}{r}\cos\frac{\pi r}{2}+\frac{x^r}{r}$ is finite when $r=0$ i.e

$\frac{B_r}{r}\cos\frac{\pi r}{2}+\frac{1}{r}+\frac{x^r-1}{r}$ is finite when $r=0$.

But $\frac{x^r-1}{r}=\log_e x$ a finite quantity when $r=0$

$\therefore \frac{B_r}{r}\cos\frac{\pi r}{2}+\frac{1}{r}$ is finite when $r=0$.

2. $1+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{x}=\Sigma\frac{1}{x}$ or $\phi(x)$

$\Sigma\frac{1}{x}=C_0+\log_e x+\frac{1}{2x}-\frac{B_2}{2x^2}+\frac{B_4}{4x^4}-\frac{B_6}{6x^6}+\frac{B_8}{8x^8}-$ &c

3. $\Sigma\frac{1}{x}=1-\frac{1}{x+1}+\frac{1}{2}-\frac{1}{x+2}+\frac{1}{3}-\frac{1}{x+3}+$ &c

$=\frac{x}{1(1+x)}+\frac{x}{2(2+x)}+\frac{x}{3(3+x)}+$ &c

4. $\Sigma\frac{1}{x}=xS_2-x^2S_3+x^3S_4-x^4S_5+x^5S_6-$ &c

5. $\Sigma\frac{1}{x-1}-\Sigma\frac{1}{-x}=-\pi\cot\pi x$.

Sol. Write πx for x in . Then we have

$\pi\cot\pi x=\frac{1}{x}-\frac{1}{1-x}+\frac{1}{1+x}-\frac{1}{2-x}+\frac{1}{2+x}-$ &c

$=\left\{1-\frac{1}{1-x}+\frac{1}{2}-\frac{1}{2-x}+\frac{1}{3}-\frac{1}{3-x}+\text{\&c}\right\}$

$-\left\{1-\frac{1}{x}+\frac{1}{2}-\frac{1}{1+x}+\frac{1}{3}-\frac{1}{2+x}+\text{\&c}\right\}$

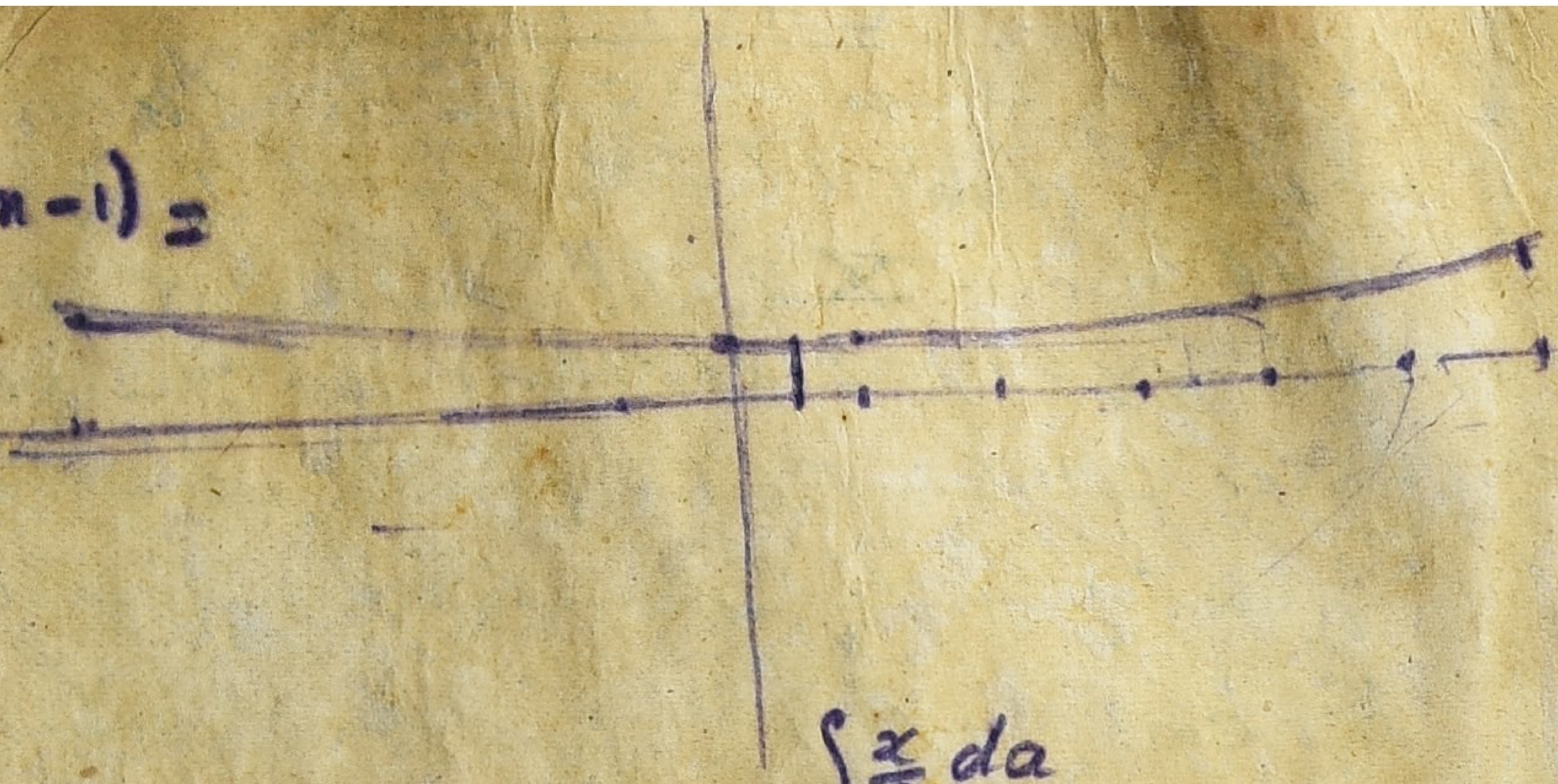
$=\Sigma\frac{1}{-x}-\Sigma\frac{1}{x-1}$ by X 3.

6. $n\,\Sigma\frac{1}{x}-\left\{\Sigma\frac{1}{x/n}+\Sigma\frac{1}{\frac{x-1}{n}}+\Sigma\frac{1}{\frac{x-2}{n}}+\ldots\Sigma\frac{1}{\frac{x-n+1}{n}}\right\}$

$=n\log_e n$.

Ex. 1. $\Sigma\frac{1}{x-\frac{1}{2}}=C_0+\log_e x+\frac{2-1}{2}\frac{B_2}{2x^2}-\frac{2^3-1}{2^3}\frac{B_4}{4x^4}+\frac{2^5-1}{2^5}\frac{B_6}{6x^6}-$ &c

$$\frac{\lfloor \frac{n}{2}}{\pi^{\frac{n}{2}}} S_n(n-1) =$$



The maximum value of $\frac{a^x}{\lfloor x} = \frac{e^{\int \frac{x}{a} da}}{\sqrt{2\pi}}$

$$= \frac{a^{a-\frac{1}{2}}}{\lfloor a-\frac{1}{2}} e^{\frac{1}{32a(36a^2+10.1)}}$$

2. $\Sigma \frac{1}{-\frac{1}{n}} + \Sigma \frac{1}{-\frac{2}{n}} + \Sigma \frac{1}{-\frac{3}{n}} + \cdots \Sigma \frac{1}{-\frac{n-1}{n}} = -n\log_e n$

3. (a) $\phi(-\frac{1}{2}) = -2\log_e 2$; (b) $\phi(-\frac{2}{3}) = -\frac{3}{2}\log_e 3 - \frac{\pi}{2\sqrt{3}}$

(c) $\phi(-\frac{3}{4}) = -\frac{\pi}{2} - 3\log_e 2$ (d) $\phi(-\frac{5}{6}) = -\frac{\pi}{2}\sqrt{3} - 2\log_e 2 - \frac{3}{2}\log_e 3$

(e) $3\phi(-\frac{1}{2}) - 2\phi(-\frac{3}{4}) = \pi$.

4. $\phi(-\frac{1}{2n}) + \phi(-\frac{3}{2n}) + \cdots \phi(-\frac{2n-1}{2n}) = -n\log_e 4n$.

7. $\frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \cdots + \frac{1}{a+xb} = \frac{1}{b}\{\phi(\frac{a}{b}+x) - \phi(\frac{a}{b})\}$

8. $\frac{1}{a+b} - \frac{1}{a+2b} + \frac{1}{a+3b} - \&c = \frac{1}{2b}\{\phi(\frac{a}{2b}) - \phi(\frac{a-b}{2b})\}$

9. $\phi(\frac{1}{2x}) = \phi(\frac{1}{x}) - \log_e 2 + x\int_0^1 \frac{n^x}{1+n^x}\,dn$.

10. $\phi(-\frac{1}{x}) = -x\int_0^1 \frac{(1-n)^2}{n(n^x-1)}\,dn$.

11. $\phi(\frac{1}{x}-1) + \phi(-\frac{1}{x}) = -x\{1 + \frac{2}{x^3-x} + \frac{2}{(2x)^3-2x} + \&c\}$

12. $\frac{2}{x^3-x} + \frac{2}{(2x)^3-2x} + \frac{2}{(3x)^3-3x} + \&c = \int_0^1 \frac{n^{x-2}(1-n)^2}{1-n^x}\,dn$

13. $1 + \frac{2}{(2x)^3-2x} + \frac{2}{(4x)^3-4x} + \frac{2}{(6x)^3-6x} + \&c$

$= \frac{1}{2}\{1 + \frac{2}{x^3-x} + \frac{2}{(2x)^3-2x} + \&c\} - \frac{\log_e 2}{x}$

+ Logmic part of $(1 - \frac{1}{1+x} + \frac{1}{1+2x} - \frac{1}{1+3x} + \&c)$

14. (a) $\frac{x}{1} - \frac{x^{1+n}}{1+n} + \frac{x^{1+2n}}{1+2n} - \&c = \int_0^x \frac{1}{1+x^n}\,dx$

(b) $\frac{x}{1} + \frac{x^{1+n}}{1+n} + \frac{x^{1+2n}}{1+2n} + \&c = \int_0^x \frac{1}{1-x^n}\,dx$

(c) If n is odd $\int_0^x \frac{1}{1-x^n}\,dx = \int_0^x \frac{1}{1+(-x)^n}\,dx$

(d) If n is even $\int_0^x \frac{1}{1-x^n}\,dx = \frac{1}{2}\int_0^x \frac{1}{1+x^{\frac{n}{2}}}\,dx + \frac{1}{2}\int_0^x \frac{1}{1-x^{\frac{n}{2}}}\,dx$

(e) If $l < n+1$

i. If n is even $\int \frac{x^{l-1}}{x^n-1}\,dx = \frac{1}{n}\log_e(x-1) + \frac{(-1)^l}{n}\log_e(x+1)$

$+ \frac{1}{n}\sum \cos\frac{rl\pi}{n}\log_e(x^2 - 2x\cos\frac{r\pi}{n} + 1)$

$\frac{2}{} \sum \sin\frac{rl\pi}{n}\tan^{-1}\frac{x-\cos\frac{r\pi}{n}}{}$ $r=2,4,$

$$\int_0^\infty \frac{x^{2m}}{(1+x^2)^{n+1}} \cos px\,dx = \frac{\pi}{2}(-1)^m \frac{e^{-p}}{2^n \lfloor n}$$

$$\times \left\{ p^n + A_1 p^{n-1} + A_2 p^{n-2} + A_3 p^{n-3} + \&c \right\}$$

$$A_r = \frac{\lfloor n+r}{\lfloor n-r} \cdot \frac{1}{2^r \lfloor r} \left\{ 1 - \frac{4}{\lfloor 1} \cdot \frac{rmn}{(n+r)(n+r-1)} \right.$$

$$\left. + \frac{4^2}{\lfloor 2} \cdot \frac{r(r-1)\,m(m-1)\,n(n-1)}{(n+r)(n+r-1)(n+r-2)(n+r-3)} - \&c \right\}$$

$$\cot\theta + 4\left(\frac{\sin 2\theta}{e^{2y}-1} + \frac{\sin 4\theta}{e^{4y}-1} + \frac{\sin 6\theta}{e^{6y}-1} + \&c \right)$$

= ~~$z\cot\phi\sqrt{1-x\sin^2\phi}$~~ $+ z\int_0^\phi \sqrt{1-x\sin^2\phi}\,d\phi.$

~~$-\frac{2\theta z}{\pi}\int_0^{\frac{\pi}{2}}$~~

$$= z\left\{ \cot\phi\sqrt{1-x\sin^2\phi} + \int_0^\phi \sqrt{1-x\sin^2\phi}\,d\phi \right.$$

$$\left. - \frac{2\theta}{\pi}\int_0^{\frac{\pi}{2}} \sqrt{1-x\sin^2\phi}\,d\phi \right\}$$

$$4\left(\frac{\sin 2\theta}{e^{y}-e^{-y}} + \frac{\sin 4\theta}{e^{2y}-e^{-2y}} + \&c \right)$$

$$= z\left\{ \int_0^\phi \sqrt{1-x\sin^2\phi}\,d\phi - \frac{2\theta}{\pi}\int_0^{\frac{\pi}{2}} \sqrt{1-x\sin^2\phi}\,d\phi \right\}$$

ii If n be odd $\int \frac{x^{l-1}}{x^n-1}dx = \frac{1}{n}\log_e(x-1) + \frac{1}{n}\sum \cos\frac{rl\pi}{n}\log_e\left(x^2-2x\cos\frac{\pi r}{n}+1\right)$

$- \frac{2}{n}\sum \sin\frac{rl\pi}{n}\tan^{-1}\frac{x-\cos\frac{r\pi}{n}}{\sin\frac{\pi r}{n}}$ $r = 2, 4, 6 \ldots (n-1)$.

iii If n be even $\int \frac{x^{l-1}}{x^n+1}dx = -\frac{1}{n}\sum \cos\frac{rl\pi}{n}\log_e\left(x^2-2x\cos\frac{\pi r}{n}+1\right)$

$+ \frac{2}{n}\sin\frac{\pi r l}{n}\tan^{-1}\frac{x-\cos\frac{\pi r}{n}}{\sin\frac{\pi r}{n}}$ $r = 1, 3, 5, \ldots (n-1)$

iv. If n be odd $\int \frac{x^{l-1}}{x^n+1}dx = \frac{(-1)^{l-1}}{n}\log_e(x+1)$

$- \frac{1}{n}\sum \cos\frac{rl\pi}{n}\log\left(x^2-2x\cos\frac{r\pi}{n}+1\right)$

$+ \frac{2}{n}\sum \sin\frac{rl\pi}{n}\tan^{-1}\frac{x-\cos\frac{r\pi}{n}}{\sin\frac{\pi r}{n}}$ $r = 1, 3, 5, \ldots (n-2)$

v. $\int_a^b f(x)\,dx = h\left\{\frac{1}{2}f(b) + \frac{1}{2}f(a) + f(a+h) + \ldots + f(b-h)\right\}$

$- \frac{h^2}{\underline{|2}}B_2\left\{f'(b) - f'(a)\right\} + \frac{h^4}{\underline{|4}}B_4\left\{f'''(b) - f'''(a)\right\} - \&c$

Ex. a. 1. $\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \&c = \log_e(1+x)$

2. $\frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \&c = \tan^{-1}x$

3. $\frac{x}{1} - \frac{x^4}{4} + \frac{x^7}{7} - \&c = \frac{1}{6}\log_e\frac{(1+x)^3}{1+x^3} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{x\sqrt{3}}{2-x}$.

4. $\frac{x}{1} - \frac{x^5}{5} + \frac{x^9}{9} - \&c = \frac{1}{4\sqrt{2}}\log_e\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}$

5. $\frac{x}{1} - \frac{x^6}{6} + \frac{x^{11}}{11} - \&c = \frac{1}{20}\log_e\frac{(1+x)^5}{1+x^5}$

$+ \frac{1}{4\sqrt{5}}\log_e\frac{1+x\cdot\frac{\sqrt{5}-1}{2}+x^2}{1-x\cdot\frac{\sqrt{5}-1}{2}+x^2} + \frac{1}{10}\sqrt{10-2\sqrt{5}}\tan^{-1}\frac{x\sqrt{10-2\sqrt{5}}}{4-x(\sqrt{5}+1)}$

$+ \frac{\sqrt{10+2\sqrt{5}}}{10}\tan^{-1}\frac{x\sqrt{10+2\sqrt{5}}}{4+x(\sqrt{5}-1)}$.

6. $\frac{x}{1} - \frac{x^7}{7} + \frac{x^{13}}{13} - \frac{x^{19}}{19} + \&c = \frac{1}{2}\tan^{-1}x + \frac{1}{6}\tan^{-1}x^3$

$+ \frac{1}{4\sqrt{3}}\log_e\frac{1+x\sqrt{3}+x^2}{1-x\sqrt{3}+x^2}$.

If $\Pi(a) = (1+a)(1+ax)(1+ax^2)$ &c ...

$$\frac{\Pi(a)\Pi(-b) - \Pi(-a)\Pi(b)}{\Pi(a)\Pi(-b) + \Pi(-a)\Pi(b)} = \frac{a-b}{1-x} + \frac{(a-bx)(ax-b)}{1-x^3} + \frac{x(a-bx^2)(ax^2-b)}{1-x^5} + \frac{x^2(a-bx^3)(ax^3-b)}{1-x^7} + \&c$$

$$\frac{(1-a^2x^4)(1-a^2x^8)(1-a^2x^{12})\&c}{(1-a^2x^2)(1-a^2x^6)(1-a^2x^{10})\&c} \cdot \frac{(1-b^2x^4)(1-b^2x^8)(1-b^2x^{12})\&c}{(1-b^2x^2)(1-b^2x^6)(1-b^2x^{10})\&c}$$

$$= \frac{1}{1-abx} + \frac{x(a-bx)(b-ax)}{(1+x^2)(1-abx)} + \frac{x(a-bx^3)(b-ax^3)}{(1+x^4)(1-abx)} +$$

$$\frac{(1-a^2x^3)(1-a^2x^7)(1-a^2x^{11})\&c}{(1-a^2x)(1-a^2x^5)(1-a^2x^9)} \cdot \frac{(1-b^2x^3)(1-b^2x^7)(1-b^2x^{11})\&c}{(1-b^2x)(1-b^2x^5)(1-b^2x^9)\&c}$$

$$= \frac{1}{1-ab} + \frac{(a-bx)(b-ax)}{(1+x^2)(1-ab)} + \frac{(a-bx^3)(b-ax^3)}{(1+x^4)(1-ab)} + \&c$$

7. $x - \frac{x^9}{9} + \frac{x^{17}}{17} - \frac{x^{25}}{25} + \&c =$

$$\frac{\sqrt{2+\sqrt{2}}}{16}\left\{\log_e \frac{1 + x\sqrt{2+\sqrt{2}} + x^2}{1 - x\sqrt{2+\sqrt{2}} + x^2} + 2\tan^{-1}\frac{x\sqrt{2+\sqrt{2}}}{1-x^2}\right\}$$

$$+ \frac{\sqrt{2-\sqrt{2}}}{16}\left\{\log_e \frac{1 + x\sqrt{2-\sqrt{2}} + x^2}{1 - x\sqrt{2-\sqrt{2}} + x^2} + 2\tan^{-1}\frac{x\sqrt{2-\sqrt{2}}}{1-x^2}\right\}$$

8. $x - \frac{x^{11}}{11} + \frac{x^{21}}{21} - \frac{x^{31}}{31} + \&c = \frac{1}{4}\tan^{-1}x - \frac{1}{20}\tan^{-1}x^5$

$$+ \frac{1}{4\sqrt{5}}\tan^{-1}\frac{(x-x^3)\sqrt{5}}{1-3x^2+x^4} + \frac{1}{40}\sqrt{10-2\sqrt{5}}\log_e \frac{1 + \frac{x}{2}\sqrt{10-2\sqrt{5}} + x^2}{1 - \frac{x}{2}\sqrt{10-2\sqrt{5}} + x^2}$$

$$+ \frac{1}{40}\sqrt{10+2\sqrt{5}}\log_e \frac{1 + \frac{x}{2}\sqrt{10+2\sqrt{5}} + x^2}{1 - \frac{x}{2}\sqrt{10+2\sqrt{5}} + x^2}$$

(L). 1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \&c = \log_e 2$

2. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c = \frac{\pi}{4}$

3. $1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \&c = \frac{\pi}{3\sqrt{3}} + \frac{1}{3}\log_e 2$

4. $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \&c = \frac{\pi}{4\sqrt{2}} + \frac{1}{2\sqrt{2}}\log_e(1+\sqrt{2})$

5. $1 - \frac{1}{6} + \frac{1}{11} - \frac{1}{16} + \&c = \frac{1}{5}\log_e 2 + \frac{1}{\sqrt{5}}\log_e\frac{1+\sqrt{5}}{2}$

$$+ \frac{\pi}{50}\left(2\sqrt{10-2\sqrt{5}} + \frac{1}{2}\sqrt{10+2\sqrt{5}}\right)$$

6. $1 - \frac{1}{7} + \frac{1}{13} - \frac{1}{19} + \&c = \frac{\pi}{6} + \frac{1}{\sqrt{3}}\log_e\frac{1+\sqrt{3}}{\sqrt{2}}$

7. $1 - \frac{1}{9} + \frac{1}{17} - \frac{1}{25} + \&c = \frac{\pi}{8\sqrt{2}}\sqrt{2+\sqrt{2}}$

$$+ \frac{1}{16}\sqrt{2+\sqrt{2}}\log_e\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}} + \frac{1}{16}\sqrt{2-\sqrt{2}}\log_e\frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}}$$

8. $1 - \frac{1}{11} + \frac{1}{21} - \frac{1}{31} + \&c = \frac{\pi}{20} +$

$$\frac{\sqrt{10-2\sqrt{5}}}{40}\log_e\frac{4+\sqrt{10-2\sqrt{5}}}{4-\sqrt{10-2\sqrt{5}}} + \frac{\sqrt{10+2\sqrt{5}}}{40}\log_e\frac{4+\sqrt{10+2\sqrt{5}}}{4-\sqrt{10+2\sqrt{5}}}$$

9. $1.\frac{1}{2} - \frac{1}{4}.\frac{1}{2^4} + \frac{1}{7}.\frac{1}{2^7} - \&c = \frac{\pi}{6\sqrt{3}} + \frac{1}{6}\log_e 3$

10. $\frac{\sqrt{3}-1}{1} - \frac{(\sqrt{3}-1)^4}{4} + \frac{(\sqrt{3}-1)^7}{7} - \&c = \frac{\pi}{4\sqrt{3}} + \frac{1}{2}\log_e\frac{1+\sqrt{3}}{\sqrt{2}}$

11. $\frac{2-\sqrt{3}}{1} - \frac{(2-\sqrt{3})^3}{5} + \frac{(2-\sqrt{3})^5}{9} - \&c = \frac{\pi}{16}(\sqrt{3}-1) - \frac{\sqrt{3}-1}{4}\log_e$ [illegible]

$$x - \frac{n}{\underline{|1}}(n+2)\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}\cdot\frac{u}{u+n+1}$$

$$+\frac{n(n+1)}{\underline{|2}}(n+4)\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\cdot\frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$$\times\frac{u(u-1)}{(u+n+1)(u+n+2)} - \&c$$

$$= n\cdot\frac{\underline{|x+n}\ \underline{|y+n}}{\underline{|n}\ \underline{|x+y+n}}\left\{1+\frac{xy}{\underline{|1}}\cdot\frac{z+u+n+1}{(z+n+1)(u+n+1)}+\right.$$

$$\left.\frac{x(x-1)\,y(y-1)\,(z+u+n+1)(z+u+n+2)}{\underline{|2}\,(z+n+1)(z+n+2)(u+n+1)(u+n+2)}+\&c\right\}$$

$$\frac{\Pi(-a)\,\Pi(-d)}{\Pi(-b)\,\Pi(-c)}\left\{1+\frac{a-b}{1-x}\cdot\frac{a-c}{a-d}+\frac{(a-b)(a-bx)(a-c)(a-cx)}{(1-x)(1-x^2)(a-d)(a-dx)}+\&c\right\}$$

$$=1+\frac{1-dx}{1-x}\cdot\frac{1-a}{a-d}\cdot\frac{b-d}{1-b}\cdot\frac{c-d}{1-c}+x^2\frac{(1-dx^3)(1-d)}{(1-x)(1-x^2)}\cdot\frac{(1-a)(1-ax)}{(a-d)(a-dx)}$$

$$\times\frac{(b-d)(b-dx)}{(1-b)(1-bx)}\cdot\frac{(c-d)(c-dx)}{(1-c)(1-cx)}+x^6\frac{(1-dx^5)(1-d)(1-dx)}{(1-x)(1-x^2)(1-x^3)}\cdot\frac{1-a}{a-d}$$

$$\times\frac{(1-ax)(1-ax^2)}{(a-dx)(a-dx^2)}\cdot\frac{(b-d)(b-dx)(b-dx^2)}{(1-b)(1-bx)(1-bx^2)}\cdot\frac{(c-d)(c-dx)(c-dx^2)}{(1-c)(1-cx)(1-cx^2)}+\&c$$

C. 1. $1 + \frac{2}{2^3-2} + \frac{2}{4^3-4} + \frac{2}{6^3-6} + \&c = 2\log_e 2$

2. $1 + \frac{2}{3^3-3} + \frac{2}{6^3-6} + \frac{2}{9^3-9} + \&c = \log_e 3$

3. $1 + \frac{2}{4^3-4} + \frac{2}{8^3-8} + \frac{2}{12^3-12} + \&c = \frac{3}{2}\log_e 2$

4. $1 + \frac{2}{5^3-5} + \frac{2}{10^3-10} + \frac{2}{15^3-15} + \&c = \frac{1}{2}\log_e 5 + \frac{1}{\sqrt{5}}\log_e \frac{\sqrt{5}+1}{2}$

5. $1 + \frac{2}{6^3-6} + \frac{2}{12^3-12} + \frac{2}{18^3-18} + \&c = \frac{1}{2}\log_e 3 + \frac{1}{3}\log_e 4$

6. $1 + \frac{2}{8^3-8} + \frac{2}{16^3-16} + \frac{2}{24^3-24} + \&c = \log_e 2 + \frac{\log_e(1+\sqrt{2})}{2\sqrt{2}}$

7. $1 + \frac{2}{10^3-10} + \frac{2}{20^3-20} + \frac{2}{30^3-30} + \&c = \frac{2}{5}\log_e 2 + \frac{1}{4}\log_e 5$

$$+ \frac{3}{2\sqrt{5}}\log_e\left(\frac{1+\sqrt{5}}{2}\right)$$

8. $1 + \frac{2}{12^3-12} + \frac{2}{24^3-24} + \frac{2}{36^3-36} + \&c = \frac{1}{2}\log_e 2 + \frac{1}{4}\log_e 3$

$$- \frac{1}{\sqrt{3}}\log_e(\sqrt{3}-1)$$

9. $1 + \frac{2}{16^3-16} + \frac{2}{32^3-32} + \frac{2}{48^3-48} + \&c = \frac{5}{8}\log_e 2 + \frac{1}{4\sqrt{2}}\log_e(1+\sqrt{2})$

$$+ \frac{\sqrt{2+\sqrt{2}}}{16}\log_e\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}} + \frac{\sqrt{2-\sqrt{2}}}{16}\log_e\frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}}$$

10. $1 + \frac{2}{20^3-20} + \frac{2}{40^3-40} + \frac{2}{60^3-60} + \&c = \frac{1}{8}\log_e 5 + \frac{3}{\sqrt{10}}$ [illegible]

$$+ \frac{3}{4\sqrt{5}}\log_e\frac{\sqrt{5}+1}{2} + \frac{\sqrt{10-2\sqrt{5}}}{40}\log_e\frac{4+\sqrt{10-2\sqrt{5}}}{4-\sqrt{10-2\sqrt{5}}}$$

$$+ \frac{\sqrt{10+2\sqrt{5}}}{40}\log_e\frac{4+\sqrt{10+2\sqrt{5}}}{4-\sqrt{10+2\sqrt{5}}}$$

15. If $\Sigma\frac{1}{x} = C_0 + \log_e a$, then

$$\left(\frac{x+\frac{1}{2}}{a}\right)^{4n} = 1 - \frac{n}{\lfloor 1}\,\frac{1}{6a^2} + \frac{n(n+1\frac{1}{10})}{\lfloor 2}\,\frac{1}{(6a^2)^2} - \frac{n(n^2+3\frac{3}{10}n+12\frac{51}{70})}{(6a^2)^3\lfloor 3} + \&c.$$

Cor. $\lfloor x$ is minimum when $x = \frac{6}{13}$ very nearly.

Sol. $\lfloor x$ is minimum when $\Sigma\frac{1}{x} = C_0$ i.e. $a = 1$

$\therefore x = \frac{1}{2} - \frac{1}{24} + \&c$ or $x = \frac{1}{2+\frac{1}{6}}$ very nearly

$$\left\{1 + a \cdot \frac{1-b}{1-x} \cdot \frac{1-c}{1-d} + a^2 \cdot \frac{(1-b)(1-bx)}{(1-x)(1-x^2)} \cdot \frac{(1-c)(1-cx)}{(1-d)(1-dx)} + \&c\right\}$$

~~$(1-a)(1-ax)(1-ax^2)$ &c~~

$$\times \left\{1 + \frac{a-b}{1-x} \cdot \frac{1-c}{1-d} + \frac{(a-b)(a-bx)}{(1-x)(1-x^2)} \cdot \frac{(1-c)(1-cx)}{(1-d)(1-dx)} + \&c\right\}$$

$$\times \frac{(1-a)(1-ax)(1-ax^2)\ \&c}{(1-b)(1-bx)(1-bx^2)\ \&c}$$

$$= 1 + \frac{a-b}{1-x} \cdot \frac{d-c}{1-d} \cdot \frac{1}{1-b}$$

$$+ \frac{(a-b)(a-bx)}{(1-x)(1-x^2)} \cdot \frac{(d-c)(dx-c)}{(1-d)(1-dx)} \cdot \frac{x}{(1-b)(1-bx)}$$

$$+ \frac{(a-b)(a-bx)(a-bx^2)}{(1-x)(1-x^2)(1-x^3)} \cdot \frac{(d-c)(dx-c)(dx^2-c)}{(1-d)(1-dx)(1-dx^2)} \cdot \frac{x^3}{(1-b)(1-bx)(1-bx^2)}$$

$$+ \&c$$

$$\frac{(1-ab)(1-abx)(1-abx^2)\ \&c}{(1-a)(1-ax)(1-ax^2)\ \&c} \cdot \frac{(1-ac)(1-acx)(1-acx^2)\ \&c}{(1-abc)(1-abcx)(1-abcx^2)\ \&c}$$

$$= 1 + a \cdot \frac{(1-b)(1-c)}{(1-a)(1-x)} + a^2 \cdot \frac{(1-b)(x-b)\ (1-c)(x-c)}{(1-a)(1-ax)\ (1-x)(1-x^2)}$$

$$+ a^3 \cdot \frac{(1-b)(x-b)(x^2-b)\ (1-c)(x-c)(x^2-c)}{(1-a)(1-ax)(1-ax^2)\ (1-x)(1-x^2)(1-x^3)} + \&c$$

16. $C_0 = \log_e 2 - 1\left(\frac{2}{3^3-3}\right) - 2\left(\frac{2}{6^3-6} + \frac{2}{9^3-9} + \frac{2}{12^3-12}\right) - \&c$

the last term in the nth group being $\frac{2}{\left(\frac{3^n+3}{2}\right)^3 - \left(\frac{3^n+3}{2}\right)}$ =

17. $\log_e \lfloor x = (x+\frac{1}{2})\log_e x - x + \frac{1}{2}\log_e(2\pi) + \frac{B_2}{1.2x} - \frac{B_4}{3.4x^3} + \&c$

Sol. Equate the coeffts. of r in IX 1.

The coefft. of r in $\frac{B_{r+1}}{r+1}\cos\frac{\pi(r+1)}{2}$ = that of r in

$= \frac{B_{r+1}}{r+1}\sin\frac{\pi r}{2}$ = that of r in $-\frac{\lfloor r}{\pi(2\pi)^r} S_{r+1}\sin\frac{\pi r}{2}$

= that of r in $-\frac{r}{2}(1 - r\log_e 2\pi + \&c)(\frac{1}{r} + C_0 - \&c)(1 - rC_0 + \&c)$

= that of r in $-\frac{1}{2} + \frac{r}{2}\log_e(2\pi) + \&c = \frac{1}{2}\log_e(2\pi)$

or as follows:-

let C be the constant in $\log_e \lfloor x$ & let $f(x) = \log_e \frac{\lfloor x}{\lfloor \frac{x}{2} \lfloor \frac{x-1}{2}}$

then $f(x) - f(x-1) = \log_e 2$. ∴ $\log_e \frac{\lfloor x}{\lfloor \frac{x}{2} \lfloor \frac{x-1}{2}} = k + x\log_e 2$.

put $x = 0$. then we see that $k = \frac{1}{2}\log_e \pi$

But the constant in $\log_e \frac{\lfloor x}{\lfloor \frac{x}{2} \lfloor \frac{x-1}{2}} = \frac{1}{2}\log_e 2 - C$.

∴ $C = \frac{1}{2}\log_e(2\pi) = .918938533204673$

Ex. Shew that when $x = \infty$ $\frac{e^x \lfloor x}{x^x\sqrt{2x+\frac{1}{3}}} = \sqrt{\pi}$.

18. $\log_e \lfloor x = -C_0 x + \frac{S_2}{2}x^2 - \frac{S_3}{3}x^3 + \frac{S_4}{4}x^4 - \&c$

i.e. $\log_e \frac{\lfloor x+2}{2} = .4227843351x + .1974670334x^2$
$- .0256856344x^3 + .0049558084x^4$
$- .0011355510x^5 + .0002863437x^6$
$- .0000766825x^7 + .0000213883x^8$
$- .0000061409x^9 + .0000054047\frac{x^{10}}{3+x}$

Ex 1. $\log_e \lfloor\frac{1}{3} = .5341990853$
$\log_e \lfloor\frac{1}{6} = .1211436313$
$\log_e \lfloor\frac{1}{12} = .0663762397$

$$\frac{1}{2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} + \frac{1}{1+\left(\frac{2}{x}\right)^2} + \cdots + \frac{1}{1+\left(\frac{x}{x}\right)^2}$$

$$= \frac{\pi x}{2}\coth \pi x - \frac{\pi x}{4} - \frac{B_2}{2x}\cdot\frac{1}{2} + \frac{B_6}{6x^5}\cdot\frac{1}{2^3} - \frac{B_{10}}{10x^9}\cdot\frac{1}{2^5} + \&c$$

$$\frac{(1-a)(1-ax)\&c\,(1-abc)(1-abcx)\&c\,(1-abd)(1-abdx)\&c\,(1-acd)\&c \times (1-acdx)\&c}{(1-ab)(1-abx)\&c\,(1-ac)(1-acx)\&c\,(1-ad)(1-adx)\&c \times (1-abcd)(1-abcdx)\&c}$$

$$= 1 - a\cdot\frac{(1-b)(1-c)(1-d)}{(1-ab)(1-ac)(1-ad)}\cdot\frac{1-ax}{1-x} + a^2\cdot\frac{(1-b)(x-b)(1-c)(x-c)}{(1-ab)(1-abx)(1-ac)(1-acx)}$$

$$\times\frac{(1-d)(x-d)}{(1-ad)(1-adx)}\cdot\frac{(1-ax^3)(1-a)}{(1-x)\,(1-x^2)} - a^3\cdot\frac{(1-b)(x-b)(x^2-b)}{(1-ab)(1-abx)(1-abx^2)}$$

$$\times\frac{(1-c)(x-c)(x^2-c)\,(1-d)(x-d)(x^2-d)}{(1-ac)(1-acx)(1-acx^2)(1-ad)(1-adx)(1-adx^2)}\cdot\frac{(1-ax^5)(1-a)(1-ax)}{(1-x)(1-x^2)(1-x^3)}$$

$$+ \&c$$

19. $\lfloor x-1 \, \lfloor -x = \pi \operatorname{cosec} \pi x.$ Cor. $\lfloor -\tfrac{1}{2} = \sqrt{\pi}$

20. $\lfloor \tfrac{x}{n} \, \lfloor \tfrac{x-1}{n} \, \lfloor \tfrac{x-2}{n} \cdots \lfloor \tfrac{x-n+1}{n} = \dfrac{(2\pi)^{\frac{n-1}{2}}}{n^{x+\frac{1}{2}}} \lfloor x$

Cor $\lfloor -\tfrac{1}{n} \, \lfloor -\tfrac{2}{n} \, \lfloor -\tfrac{3}{n} \, \lfloor -\tfrac{4}{n} \cdots \lfloor -\tfrac{n-1}{n} = \dfrac{(2\pi)^{\frac{n}{2}}}{\sqrt{2\pi n}}$

Ex. 1. $(\lfloor -\tfrac{1}{3})^2 = \lfloor -\tfrac{1}{6} \sqrt{\tfrac{\pi}{3}} \sqrt[3]{4}$

2. $\lfloor i \, \lfloor -i = \dfrac{2\pi e^{\pi}}{e^{2\pi}-1}$

3. $2^i \lfloor \tfrac{i}{2} \, \lfloor \tfrac{i-1}{2} = \sqrt{\pi} \, \lfloor i$

4. $\dfrac{\lfloor x}{\lfloor \frac{x}{2} \, \lfloor \frac{x-1}{2}} = \dfrac{2^x}{\sqrt{\pi}}$

5. $\log_e \lfloor x-\tfrac{1}{2} = x\log_e x - x + \log_e \sqrt{2\pi} + \dfrac{2-1}{2} \cdot \dfrac{B_2}{1 \cdot 2 x} - \dfrac{2^3-1}{2^3} \cdot \dfrac{B_4}{3 \cdot 4 x^3} + \&c$

21. $\dfrac{\log_e 1}{1} + \dfrac{\log_e 2}{2} + \cdots + \dfrac{\log_e x}{x} = \phi(x)$

$\phi(x) = \left(\Sigma \tfrac{1}{x} - C_0\right)\log_e x - \tfrac{1}{2}(\log_e x)^2 + C_1 + \dfrac{B_2}{2x^2} \cdot 1 - \dfrac{B_4}{4x^4}\left(1+\tfrac{1}{2}+\tfrac{1}{3}\right)$

$+ \dfrac{B_6}{6x^6}\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}\right) - \&c$

$C_1 = -\cdot 072815845483680$

Sol. Write $x-1$ for x in IX 1, then divide both sides by x^2 and find the coeffts. of x from both sides & equate them.

Ex when $x = \infty$ show that $\phi(x) - \tfrac{1}{2}\left(\Sigma\tfrac{1}{x} - C_0\right)^2 = 0$.

22. $\phi(x) = \dfrac{\log_e 1}{1} - \dfrac{\log_e(1+x)}{1+x} + \dfrac{\log_e 2}{2} - \dfrac{\log_e(2+x)}{2+x} + \&c$

Cor $\dfrac{\log_e 1}{1} - \dfrac{\log_e 3}{3} + \dfrac{\log_e 5}{5} - \dfrac{\log_e 7}{7} + \&c = \dfrac{\pi}{4}\log_e 2 + \tfrac{1}{4}\left\{\phi(-\tfrac{1}{4}) - \phi(\tfrac{3}{4})\right\}.$

23. $n\phi(x) - \left\{\phi\left(\tfrac{x}{n}\right) + \phi\left(\tfrac{x-1}{n}\right) + \cdots + \phi\left(\tfrac{x-n+1}{n}\right)\right\}$

$= n\log_e n\left(\Sigma\tfrac{1}{x} - C_0\right) - \tfrac{n}{2}(\log_e n)^2$

$$\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots = \phi(x)$$

$$\phi\left(\frac{x}{1-y}\right) + \phi\left(\frac{y}{1-x}\right)$$
$$= \phi(x) + \phi(y) + \phi\left\{\frac{xy}{(1-x)(1-y)}\right\} + \log(1-x)\log(1-y).$$

$$\frac{1 + \frac{a-b}{1-x}\cdot\frac{1-c}{1-d} + \frac{(a-b)(a-bx)}{(1-x)(1-x^2)}\cdot\frac{(1-c)(1-cx)}{(1-d)(1-dx)} + \&c}{1 + \frac{c-d}{1-x}\cdot\frac{1-a}{1-b} + \frac{(c-d)(c-dx)}{(1-x)(1-x^2)}\cdot\frac{(1-a)(1-ax)}{(1-b)(1-bx)} + \&c}$$

$$= \frac{(1-b)(1-bx)(1-bx^2)\,\&c}{(1-a)(1-ax)(1-ax^2)\,\&c}\cdot\frac{(1-c)(1-cx)(1-cx^2)\,\&c}{(1-d)(1-dx)(1-dx^2)\,\&c}$$

$$\int_0^\infty \left\{\frac{n^x\, n!}{(n+x)!} + e^{-x}\left(1+\frac{x}{n}\right)^n\right\} dx = \frac{e^n\, n!}{n^n} + \frac{6n}{12n+1}$$

very very nearly.

$$1 + \frac{n}{1!} + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\,\theta = \frac{e^n}{2}.$$

where $\theta = \frac{4+15n}{8+45n}$ very nearly.

~~or still more approximately~~

~~$\theta = \frac{2+7nu}{4+21nu}$. where $u = \frac{466+585n}{480+546n}$.~~

Cor. $\phi\left(-\frac{1}{n}\right) + \phi\left(-\frac{2}{n}\right) + \phi\left(-\frac{3}{n}\right) + \cdots + \phi\left(-\frac{n-1}{n}\right)$

$$= n C_0 \log_e n + \frac{n}{2}(\log_e n)^2$$

Ex. 1. $\frac{\sqrt[1]{1}}{\sqrt[2]{2}} \cdot \frac{\sqrt[3]{3}}{\sqrt[4]{4}} \cdot \frac{\sqrt[5]{5}}{\sqrt[6]{6}} \cdot \frac{\sqrt[7]{7}}{\sqrt[8]{8}} \cdot \frac{\sqrt[9]{9}}{\sqrt[10]{10}} \cdots$ ad inf. $= 2^{\frac{1}{2}\log_e 2 - C_0}$

2. $\phi\left(-\frac{1}{2}\right) = (\log_e 2)^2 + 2C_0 \log_e 2$

3. $\phi\left(-\frac{1}{3}\right) + \phi\left(-\frac{2}{3}\right) = \frac{3}{2}(\log_e 3)^2 + 3C_0 \log_e 3.$

4. $\phi\left(-\frac{1}{4}\right) + \phi\left(-\frac{3}{4}\right) = 7(\log_e 2)^2 + 6C_0 \log_e 2.$

5. $\phi\left(-\frac{1}{6}\right) + \phi\left(-\frac{5}{6}\right) = C_0(3\log_e 3 + 4\log_e 2) + \frac{3}{2}(\log_e 12)^2 - (\log_e 2)^2.$

24. $\frac{\pi}{2}\left\{\log_e \frac{\lfloor x-1}{\lfloor -x} + (C_0 + \log_e 2\pi)(2x-1)\right\}$ when x lies between 0 & 1

$$= \frac{\log_e 1}{1}\sin 2\pi x + \frac{\log_e 2}{2}\sin 4\pi x + \frac{\log_e 3}{3}\sin 6\pi x + \&c.$$

N. B. $\frac{\pi}{2} - \pi x = \sin 2\pi x + \frac{1}{2}\sin 4\pi x + \frac{1}{3}\sin 6\pi x + \&c$ for the same limits

25. $\phi(x-1) - \phi(-x) = (C_0 + \log_e 2\pi)\,\pi \cot \pi x$

$$+ 2\pi\left\{\sin 2\pi x \log_e 1 + \sin 4\pi x \log_e 2 + \&c\right\}$$

N. B. $\sin 2\pi x + \sin 4\pi x + \sin 6\pi x + \&c = \frac{1}{2}\cot \pi x.$

Ex. 1. Find the values of $\phi\left(-\frac{1}{2}\right)$, $\phi\left(-\frac{1}{3}\right)$, $\phi\left(-\frac{3}{4}\right)$ & $\phi\left(-\frac{5}{6}\right)$.

2. $\frac{\log_e 1}{1} - \frac{\log_e 3}{3} + \frac{\log_e 5}{5} - \frac{\log_e 7}{7} + \&c = \frac{\pi}{4}\log_e \pi - \pi \log_e \lfloor -\tfrac{1}{4} - \frac{\pi}{4}C_0.$

3. $$\frac{\left(\frac{\sqrt[1]{1}}{\sqrt[2]{2}} \cdot \frac{\sqrt[3]{3}}{\sqrt[4]{4}} \cdot \frac{\sqrt[5]{5}}{\sqrt[6]{6}} \cdot \frac{\sqrt[7]{7}}{\sqrt[8]{8}} \cdots\right)^{\frac{1}{\log_e 2}}}{\left(\frac{\sqrt[1]{1}}{\sqrt[3]{3}} \cdot \frac{\sqrt[5]{5}}{\sqrt[7]{7}} \cdot \frac{\sqrt[9]{9}}{\sqrt[11]{11}} \cdot \frac{\sqrt[13]{13}}{\sqrt[15]{15}} \cdots\right)^{\frac{4}{\pi}}} = \frac{\sqrt{2}}{\pi}\left(\lfloor -\tfrac{1}{4}\right)^4$$

26. $(\log_e 1)^2 + (\log_e 2)^2 + (\log_e 3)^2 + \cdots + (\log_e x)^2 = \psi(x)$

$$\psi(x) = 2\log_e x \log_e \frac{\lfloor x}{\sqrt{2\pi}} - \left(x + \tfrac{1}{2}\right)(\log_e x)^2 + 2x + \tfrac{1}{2}C_0^2 + C_1 - \frac{\pi^2}{24} - \tfrac{1}{2}(\log_e 2\pi)^2$$

$$+ 2\left\{\frac{B_4}{3.4} \cdot \frac{1 + \frac{1}{2}}{x^3} - \frac{B_6}{5.6} \cdot \frac{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}}{x^5} + \&c\right\}$$

Sol. Equate the coeff$^{ts.}$ of n^2 in IX.

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* $1 + \frac{\beta x}{(1-x)(1-\alpha x)} + \frac{\beta^2 x^4}{(1-x)(1-x^2)(1-\alpha x)(1-\alpha x^2)} + \&c$

$\times (1-\alpha x)(1-\alpha x^2)(1-\alpha x^3)$ &c

$= 1 - x\frac{\alpha-\beta}{1-x} + x^3 \cdot \frac{(\alpha-\beta)(\alpha-\beta x)}{(1-x)(1-x^2)}$

$- x^6 \cdot \frac{(\alpha-\beta)(\alpha-\beta x)(\alpha-\beta x^2)}{(1-x)(1-x^2)(1-x^3)} + \&c$ ~~$= F(\beta)$~~

$$\frac{F(\beta)}{F(\beta x)} = 1 + \cfrac{\beta x}{1-\alpha x + \cfrac{\beta x^2}{1-\alpha x^2 + \cfrac{\beta x^3}{1-\alpha x^3 + \&c}}}$$

where * $= F(\beta)$.

$$= \frac{N}{D}.$$

$$\cfrac{1}{1+\cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\cdots+\cfrac{a_{n+2}}{1}}}}}$$

$D = \phi_0(n) + \phi_1(n) + \phi_2(n) + \&c.$

& N is the D of $\cfrac{1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\cdots\cfrac{a_{n+2}}{1}}}}$

27. $\psi(x) - \left\{ \psi\left(\frac{x}{n}\right) + \psi\left(\frac{x-1}{n}\right) + \psi\left(\frac{x-2}{n}\right) + \dots + \psi\left(\frac{x-n+1}{n}\right) \right\}$

$$= 2\log_e n \log_e \frac{\lfloor x}{\sqrt{2\pi}} - x(\log_e n)^2 - (n-1)\left\{\frac12 C_0^2 + C_1 - \frac{\pi^2}{24} - \frac12(\log_e 2\pi)^2\right\} - \frac12(\log_e n)^2$$

Cor. $\psi\left(-\frac1n\right) + \psi\left(-\frac2n\right) + \dots + \psi\left(-\frac{n-1}{n}\right)$

$$= \log_e n \log_e 2\pi + (n-1)\left\{\frac12 C_0^2 + C_1 - \frac{\pi^2}{24} - \frac12(\log_e 2\pi)^2\right\} + \frac12(\log_e n)^2$$

Ex. 1. $\dfrac{1^{\frac11} \cdot 2^{\frac12} \cdot 3^{\frac13} \cdot 4^{\frac14} \dots x^{\frac1x}}{1^{\log_e 1} \cdot 2^{\log_e 2} \dots x^{\log_e x}} \, x^{x\log_e x - 2x} \, e^{2x + \frac12\left(\Sigma\frac1x - \log_e x\right)^2}$

$$= e^{\frac{\pi^2}{24}} (2\pi)^{\log_e \sqrt{2\pi}} \text{ when } x = \infty.$$

2. $\psi\left(-\frac12\right) = \frac12 C_0^2 + C_1 - \frac{\pi^2}{24} - \frac12\left\{(\log_e 2)^2 + (\log_e \pi)^2\right\} + (\log_e 2)^2$

3. $\psi\left(-\frac13\right) + \psi\left(-\frac23\right) = C_0^2 + 2C_1 - \frac{\pi^2}{12} - (\log_e 2\pi)^2 + \log_e 3 \log_e 2\pi + \frac12(\log_e 3)^2$

4. $\psi\left(-\frac14\right) + \psi\left(-\frac34\right) = C_0^2 + 2C_1 - \frac{\pi^2}{12} - \log_e \pi \log_e 2\pi + 1\frac12(\log_e 2)^2$

5. $\frac12\left\{\psi\left(-\frac16\right) + \psi\left(-\frac56\right)\right\} = \frac12 C_0^2 + C_1 - \frac{\pi^2}{24} - \frac12(\log_e 2\pi) + \frac12 \log_e 2 \log_e 3.$

28. $\dfrac{\psi(x-1) + \psi(-x)}{2} = C_1 - \dfrac{\pi^2}{24} + \dfrac12(C_0 + \log_e 2\pi)\left(C_0 - \log_e \dfrac{\pi}{2\sin^2 \pi x}\right)$

$$- \left\{\frac{\log_e 1}{1}\cos 2\pi x + \frac{\log_e 2}{2}\cos 4\pi x + \&c\right\}$$

29. If C_n be the constant in $(\log_e 1)^n + (\log_e 2)^n + \dots (\log_e x)^n$ and if $\phi_n(x) = (\log_e 1)^n + (\log_e 2)^n + \dots + (\log_e x)^n - C_n$, then

i. The logarithmic part of $\phi_n(x) = n \log_e x \, \phi_{n-1}(x)$

$$- \frac{n(n-1)}{\lfloor 2}(\log_e x)^2 \phi_{n-2}(x) + \frac{n(n-1)(n-2)}{\lfloor 3}(\log_e x)^3 \phi_{n-3}(x) - \&c.$$

and the non-logarithmic part can be found from IX 1.

ii. $\phi_0(x)(\log_e x)^n - \frac{n}{\lfloor 1}\phi_1(x)(\log_e x)^{n-1} + \frac{n(n-1)}{\lfloor 2}\phi_2(x)(\log_e x)^{n-2} - \&c$

$$= x\lfloor n - \frac{1}{x^n} \cdot \frac{B_{n+1}}{n+1}\sin\frac{\pi n}{2} - \frac{n}{2} \cdot \frac{1}{x^{n+1}} \frac{B_{n+2}}{n+2}\cos\frac{\pi n}{2}$$

$$+ \frac{n\left(n+1\frac23\right)}{2 \cdot 4} \cdot \frac{1}{x^{n+2}} \cdot \frac{B_{n+3}}{n+3}\sin\frac{\pi n}{2} + \frac{n(n+2)(n+3)}{2 \cdot 4 \cdot 6} \dots B$$

$$\times \cos\frac{\pi n}{2}$$

$$1 + \frac{x^2}{(1-x)^2} + \frac{x^6}{(1-x)^2(1-x^4)^2} + \frac{x^{12}}{(1-x)^2(1-x^4)^2(1-x^6)^2} + \&c$$

$$= \frac{1 - x + x^3 - x^6 + x^{10} - \&c}{(1-x)(1-x^2)(1-x^3)(1-x^4)\ \&c}$$

$$\frac{(1+xy)(1+x^3y)(1+x^5y)\ \&c}{(1+\alpha xy)(1+\alpha x^3y)(1+\alpha x^5y)\ \&c} \cdot \frac{(1+\frac{x}{y})(1+\frac{x^3}{y})(1+\frac{x^5}{y})\ \&c}{(1+\beta.\frac{x}{y})(1+\beta\frac{x^3}{y})(1+\beta\frac{x^5}{y})\ \&c}$$

$$\times \frac{(1-x^2)(1-x^4)(1-x^6)\ \&c \, . \, (1-\alpha\beta x^2)(1-\alpha\beta x^4)(1-\alpha\beta x^6)\ \&c}{(1-\alpha x^2)(1-\alpha x^4)(1-\alpha x^6)\ \&c \, . \, (1-\beta x^2)(1-\beta x^4)(1-\beta x^6)\ \&c}.$$

$$= 1 + \left\{xy . \frac{1-\alpha}{1-\beta x^2} + \frac{x}{y} . \frac{1-\beta}{1-\alpha x^2}\right\}$$

$$+ \left\{(xy)^2 . \frac{(1-\alpha)(x^2-\alpha)}{(1-\beta x^2)(1-\beta x^4)} + \left(\frac{x}{y}\right)^2 \frac{(1-\beta)(x^2-\beta)}{(1-\alpha x^2)(1-\alpha x^4)}\right\}$$

$$+ \left\{(xy)^3 . \frac{(1-\alpha)(x^2-\alpha)(x^4-\alpha)}{(1-\beta x^2)(1-\beta x^4)(1-\beta x^6)} + \left(\frac{x}{y}\right)^3 \frac{(1-\beta)(x^2-\beta)(x^4-\beta)}{(1-\alpha x^2)(1-\alpha x^4)(1-\alpha x^6)}\right\}$$

$+ \&c \quad \&c \quad \&c$

$$- \frac{n(n+2)(n+4)^2 + \frac{n(n+2)}{3} + \frac{4n}{5}}{2\cdot4\cdot6\cdot8} \cdot \frac{1}{x^{n+4}} \cdot \frac{B_{n+5}}{n+5} \sin\frac{\pi n}{2}$$

$$- \frac{n(n+4)(n+5)\left\{(n+3)(n+4)+\frac{2}{3}(n+1)\right\}}{2\cdot4\cdot6\cdot8\cdot10} \cdot \frac{1}{x^{n+5}} \cdot \frac{B_{n+6}}{n+6} \cos\frac{\pi n}{2} + \&c.$$

iii. $\phi_n\left(\frac{x}{r}\right) + \phi_n\left(\frac{x-1}{r}\right) + \cdots + \phi_n\left(\frac{x-r+1}{r}\right)$

$= \phi_n(x) - n \log_e r \, \phi_{n-1}(x) + \frac{n(n-1)}{\lfloor 2}(\log_e r)^2 \phi_{n-2}(x) - \&c$

Cor 1. $\phi_n\left(-\frac{1}{r}\right) + \phi_n\left(-\frac{2}{r}\right) + \cdots + \phi_n\left(-\frac{r-1}{r}\right)$

$= -\left\{C_n - n \log_e r \, C_{n-1} + \frac{n(n-1)}{\lfloor 2}(\log_e r)^2 C_{n-2} - \&c\right\}$

Cor 2. There will be no logarithmic function in

$\phi_n\left(\frac{1}{x}\right) + \phi_n\left(\frac{2}{x}\right) + \cdots + \phi_n\left(\frac{x}{x}\right)$

30. Let $1^n + 2^n k + 3^n k^2 + \cdots + x^n k^{x-1} = k^x \phi(x) = F_k(x)$, then

i. $\phi(x) = C_n(k) + x^n \frac{\psi_0(-k)}{k-1} - \frac{n}{\lfloor 1} x^{n-1} \frac{\psi_1(-k)}{(k-1)^2} + \frac{n(n-1)}{\lfloor 2} \cdot \frac{\psi_2(-k)}{(k-1)^3} - \&c$ where ψ is the same ψ in

ii. $C_n(k) = \frac{\psi_n(-k)}{(1-k)^{n+1}}$ &c $k\psi_n(-k) = k^n \psi\left(-\frac{1}{k}\right)$.

iii. $F_k\left(\frac{x}{n}\right) + F_k\left(\frac{x-1}{n}\right) + \cdots + F_k\left(\frac{x-n+1}{n}\right) - n\, C_n(k)$

$= \frac{\sqrt[n]{k}}{k\, n^n}\left\{F_{\sqrt[n]{k}}(x) - C_n(\sqrt[n]{k})\right\}$

Cor. $F_k\left(-\frac{1}{n}\right) + F_k\left(-\frac{2}{n}\right) + \cdots + F_k\left(-\frac{n-1}{n}\right) = n\, C_n(k) - \frac{\sqrt[n]{k}\, C_n(\sqrt[n]{k})}{k\, n^n}$.

31. Let $\frac{\log_e 1}{1^n} + \frac{\log_e 2}{2^n} + \frac{\log_e 3}{3^n} + \cdots + \frac{\log_e x}{x^n} = \phi_n(x)$ & let C_n' be the constant. Then

i. $\phi_n(x) = C_n' - \left\{\frac{1}{(x+1)^n} + \frac{1}{(x+2)^n} + \frac{1}{(x+3)^n} + \&c\right\} \log_e x - \frac{1}{(n-1)^2} \frac{1}{x^{n-1}}$

$+ B_2 \frac{n}{\lfloor 2} \cdot \frac{1}{n x^{n+1}} - B_4 \frac{n(n+1)(n+2)}{\lfloor 4}\left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}\right)\frac{1}{x^{n+3}}$

$+ B_6 \frac{n(n+1)(n+2)(n+3)(n+4)}{\lfloor 6}\left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4}\right)\frac{1}{x^{n+5}} - \&c$

If $\int_{\alpha_1}^{\beta_1} \phi_1(p, x)\, F(nx)\, dx = \psi_1(p, n)$

& $\int_{\alpha_2}^{\beta_2} \phi_2(p, x)\, F(nx)\, dx = \psi_2(p, n)$

then $\int_{\alpha_1}^{\beta_1} \phi_1(p, x)\, \psi_2(q, nx)\, dx = \int_{\alpha_2}^{\beta_2} \phi_2(q, x)\, \psi_1(p, x)$

$$\int_{-\infty}^{\infty} \frac{e^{mx} \frac{\cos}{\sin}(px)}{(1+e^x)^n}\, dx$$

$$= \frac{\lfloor m-1 \; \lfloor n-m-1}{\lfloor n-1} \cdot \frac{\cos}{\sin}\left(\tan^{-1}\frac{p}{n-m} - \tan^{-1}\frac{p}{m} + \tan^{-1}\frac{p}{n-m+1} - \tan^{-1}\frac{p}{m+1} + \&c\right)$$

$$\div \sqrt{\left\{1+\frac{p^2}{(n-m)^2}\right\}\left\{1+\frac{p^2}{m^2}\right\}\left\{1+\frac{p^2}{(n-m+1)^2}\right\}\left\{1+\frac{p^2}{(m+1)^2}\right\}\&c}$$

$$\int_0^{\infty} e^{-mx}(1-e^{-x})^n \frac{\cos}{\sin}(px)\, dx$$

ii. $\phi_r(x) = r x C'_{r+1} - \frac{r(r+1)}{\lfloor 2} x^2 C'_{r+2} + \frac{r(r+1)(r+2)}{\lfloor 3} x^3 C'_{r+3} - \&c$

$- r \cdot \frac{1}{r} x S_{r+1} + \frac{r(r+1)}{\lfloor 2}\left(\frac{1}{r} + \frac{1}{r+1}\right) x^2 S_{r+2} - \&c.$

iii. $n^r \phi_r(x) - \left\{\phi_r\left(\frac{x}{n}\right) + \phi_r\left(\frac{x-1}{n}\right) + \cdots + \phi_r\left(\frac{x-n+1}{n}\right)\right\}$

$= C'_r(n^r - n) - n^r \log_e n \left\{\frac{1}{(x+1)^r} + \frac{1}{(x+2)^r} + \frac{1}{(x+3)^r} + \&c\right\}$

Cor. $\phi_r\left(-\frac{1}{n}\right) + \phi_r\left(-\frac{2}{n}\right) + \phi_r\left(-\frac{3}{n}\right) + \cdots + \phi_r\left(-\frac{n-1}{n}\right)$

$= n^r \log_e n\, S_r - (n^r - n) C'_r.$

32. Let $(\log_e 1)^r + \frac{1}{2}(\log_e 2)^r + \frac{1}{3}(\log_e 3)^r + \&c$ to x terms $= \phi_r(x)$ & let C_r be its constant, then

i. $\phi_r(x) - \frac{1}{r+1}(\log_e x)^{r+1} = C_r$ when $x = \infty$

ii. $n\,\phi_r(x) - \left\{\phi_r\left(\frac{x}{n}\right) + \phi_r\left(\frac{x-1}{n}\right) + \phi_r\left(\frac{x-2}{n}\right) + \cdots + \phi_r\left(\frac{x-n+1}{n}\right)\right\}$

$= \frac{n}{r+1}(\log_e n)^{r+1} \cos \pi r + r n \log_e n \left\{\phi_{r-1}(x) - C_{r-1}\right\}$

$\frac{r(r-1)}{\lfloor 2} n(\log_e n)^2 \left\{\phi_{r-2}(x) - C_{r-2}\right\} + \&c$ the last term being

$(-1)^{r-1} n(\log_e n)^r \left\{\phi_0(x) - C_0\right\}$

33. $\frac{(\log_e 1)^r}{1^{r+1}} + \frac{(\log_e 2)^r}{2^{r+1}} + \frac{(\log_e 3)^r}{3^{r+1}} + \&c$

$= \frac{\lfloor r}{n^{r+1}} + C_r - \frac{n}{\lfloor 1} C_{r+1} + \frac{n^2}{\lfloor 2} C_{r+2} - \frac{n^3}{\lfloor 3} C_{r+3} + \&c$

Sol. Differentiate r times both sides in

Ex. Show that

1. $\frac{(\log_e 1)^3}{1\sqrt{1}} + \frac{(\log_e 2)^3}{2\sqrt{2}} + \frac{(\log_e 3)^3}{3\sqrt{3}} + \&c = 96.001$ nearly.

2. $\frac{\log_e 1}{1^2} + \frac{\log_e 2}{2^2} + \frac{\log_e 3}{3^2} + \&c = \cdot 9382$ nearly.

3. $\frac{(\log_e 1)^4}{1^2} + \frac{(\log_e 2)^4}{2^2} + \frac{(\log_e 3)^4}{3^2} + \&c = 24$ nearly.

4. $\frac{(\log_e 1)^5}{1\sqrt{1}} + \frac{(\log_e 2)^5}{2\sqrt{2}} + \frac{(\log_e 3)^5}{3\sqrt{3}} + \&c = 7680$ nearly.

If $\int_0^\infty \phi_1(x)\, F(nx)\, dx = \psi_1(n)$

& $\int_0^\infty \phi_2(x)\, F(nx)\, dx = \psi_2(n)$

then $\int_0^\infty \phi_1(ax)\, \psi_2(bx)\, dx = \int_0^\infty \phi_2(ax)\, \psi_1(bx)$

If $\int_{\alpha_1}^{\beta_1} \phi_1(x)\, F(nx)\, dx = \psi_1(n)$
& $\int_{\alpha_2}^{\beta_2} \phi_2(x)\, F(nx)\, dx = \psi_2(n)$ } then

$$\int_{\alpha_1}^{\beta_1} \phi_1(x)\, \psi_2(nx)\, dx = \int_{\alpha_2}^{\beta_2} \phi_2(x)\, \psi_1(nx)\, dx$$

5. $\frac{(\log_e 1)^5}{1^2}\sqrt{\log_e 1} + \frac{(\log_e 2)^5}{2^2}\sqrt{\log_e 2} + \frac{(\log_e 3)^5}{3^2}\sqrt{\log_e 3} + \&c = 2.88$ nearly.

34. $\frac{\log_e 1}{\sqrt{1}} + \frac{\log_e 2}{\sqrt{2}} + \frac{\log_e 3}{\sqrt{3}} + \cdots + \frac{\log_e x}{\sqrt{x}} = \phi(x)$

i $\phi(x) = \frac{\log_e 1}{\sqrt{1}} - \frac{\log_e(1+x)}{\sqrt{1+x}} + \frac{\log_e 2}{\sqrt{2}} - \frac{\log_e(2+x)}{\sqrt{2+x}} + \&c$

ii. $\phi(x) = \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{x}}\right)\log_e x$

$+ (\sqrt{2}+1)\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \&c\right)\left(\log_e x + \frac{1}{2}C_0 + \frac{\pi}{4} + \log_e\sqrt{8\pi}\right)$

$- 4\sqrt{x} + \frac{1}{2}\cdot\frac{B_2}{x\sqrt{x}} - \frac{1.3.5}{2.4.6}\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{B_4}{2x^3\sqrt{x}}$

$+ \frac{1.3.5.7.9}{2.4.6.8.10}\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}\right)\frac{B_6}{3x^5\sqrt{x}} - \&c$

iii. $\phi(x) - \frac{1}{\sqrt{n}}\left\{\phi\left(\frac{x}{n}\right) + \phi\left(\frac{x-1}{n}\right) + \cdots + \phi\left(\frac{x-n+1}{n}\right)\right\}$

$= \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{x}}\right)\log_e n$

$- (1+\sqrt{2})\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \&c\right)\left\{(\sqrt{n}-1)\left(\frac{1}{2}C_0 + \frac{\pi}{4} + \log_e\sqrt{8\pi}\right) - \log_e n\right\}$

iv. If $\psi(x) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{x}}$, then

$\left\{\phi(x-1) + \phi(-x) - 2C\right\} + \left(C_0 + \frac{\pi}{2} + \log_e 8\pi\right)\left\{\psi(x-1) + \psi(-x) - 2C'\right\}$

$= 2\left\{\frac{\log_e 1}{\sqrt{1}}\cos 2\pi x + \frac{\log_e 2}{\sqrt{2}}\cos 4\pi x + \&c\right\}$

v. $\left\{\phi(x-1) - \phi(-x)\right\} + \left(C_0 - \frac{\pi}{2} + \log_e 8\pi\right)\left\{\psi(x-1) - \psi(-x)\right\}$

$= 2\left\{\frac{\log_e 1}{\sqrt{1}}\sin 2\pi x + \frac{\log_e 2}{\sqrt{2}}\sin 4\pi x + \&c\right\}$

In both cases C & C' are the constants of $\phi(x)$ & $\psi(x)$ respectively.

Ex. 1. Find the values of $\phi(-\frac{1}{2})$, $\phi(-\frac{3}{2})$ & $\phi(-\frac{3}{4})$.

2. Show that the constant in $\phi(x) = -\frac{1}{2}S_{\frac{1}{2}}\left(C_0 + \frac{\pi}{2} + \log_e 8\pi\right)$

$= 3.92265\ldots = 2\left\{2 - \frac{1}{2}\cdot\frac{B_2}{2} + \frac{1.3.5}{2.4.6}\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{B_4}{4} - \&c\right\}$

Sol. Write $\frac{1+h}{2}$ for n in IX 4. and equate the coeff^{ts} of h.

Put $x = 1$ in X 34 ii. then the 2nd result is at once obtained.

$\phi(1) + \phi(2) + \cdots + \phi(h)$

$= \int^{h} \phi(x)\,dx + \frac{1}{2}\phi(h) + \int_0^\infty \frac{\phi(h+xi) - \phi(h-xi)}{i(e^{2\pi x} - 1)}\,dx$

The imaginary part in

$\phi(n)\,F(a+bi) - \phi(2n)\,F(a+2bi) + \phi(3n)\,F(a+3bi) - \&c$

$= \int_0^\infty \frac{F(a+bx) - F(a-bx)}{e^{\pi x} - e^{-\pi x}}\,\phi(n^2x^2)\,dx.$

$\int_0^\infty \frac{e^{ax} - e^{-ax}}{e^{\pi x} - e^{-\pi x}} \cdot \frac{dx}{1+n^2x^2}$

$= \frac{\sin a}{1+n} - \frac{\sin 2a}{1+2n} + \frac{\sin 3a}{1+3n} - \&c$

a lying between 0 & π.

$\operatorname{cosec}\theta + 4\left\{\frac{\sin\theta}{e^{y}-1} + \frac{\sin 3\theta}{e^{3y}-1} + \frac{\sin 5\theta}{e^{5y}-1}\right\}$

$= 2\operatorname{cosec}\phi$

$\frac{\cos\theta}{\sinh\frac{y}{2}} - \frac{\cos 3\theta}{\sinh\frac{3y}{2}} + \frac{\cos 5\theta}{\sinh\frac{5y}{2}} - \&c$

$= \frac{z}{2}\sqrt{z}\,.\,\frac{\cos\phi}{\sqrt{1-x\sin^2\phi}}$

CHAPTER I

1. If $S_n = \frac{1}{(1-a)^n} - \frac{1}{(1+a)^n} + \frac{1}{(3-a)^n} - \frac{1}{(3+a)^n} + \&c$, then

i. If n is odd,

$$\frac{\cos(1-a)x}{(1-a)^n} - \frac{\cos(1+a)x}{(1+a)^n} + \frac{\cos(3-a)x}{(3-a)^n} - \frac{\cos(3+a)x}{(3+a)^n} + \&c$$

$= S_n - \frac{x^2}{2!} S_{n-2} + \frac{x^4}{4!} S_{n-4} - \&c$ as far as the term containing S_1

ii. If n is even,

$$\frac{\sin(1-a)x}{(1-a)^n} - \frac{\sin(1+a)x}{(1+a)^n} + \frac{\sin(3-a)x}{(3-a)^n} - \frac{\sin(3+a)x}{(3+a)^n} + \&c$$

$= \frac{x}{1!} S_{n-1} - \frac{x^3}{3!} S_{n-3} + \frac{x^5}{5!} S_{n-5} - \&c$ as far as the term containing S_1

2. If $S_n = \frac{1}{(1-a)^n} + \frac{1}{(1+a)^n} + \frac{1}{(3-a)^n} + \frac{1}{(3+a)^n} + \&c$, then

i. If n is even

$$\frac{\cos(1-a)x}{(1-a)^n} + \frac{\cos(1+a)x}{(1+a)^n} + \frac{\cos(3-a)x}{(3-a)^n} + \frac{\cos(3+a)x}{(3+a)^n} + \&c$$

$= S_n - \frac{x^2}{2!} S_{n-2} + \frac{x^4}{4!} S_{n-4} - \&c$ as far as the term containing S_2

ii. If n is odd

$$-\frac{\sin(1-a)x}{(1-a)^n} + \frac{\sin(1+a)x}{(1+a)^n} + \frac{\sin(3-a)x}{(3-a)^n} + \frac{\sin(3+a)x}{(3+a)^n} + \&c$$

$= \frac{x}{1!} S_{n-1} - \frac{x^3}{3!} S_{n-3} + \frac{x^5}{5!} S_{n-5} - \&c$ as far as the term containing S_2

Sol. In both 1 & 2 expand the series in ascending powers of x and apply

3. If $\phi(n) = \frac{\cos x}{1^n} - \left(1+\frac{1}{2}\right)\frac{\cos 3x}{3^n} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{\cos 5x}{5^n} - \&c$ then

$$\int_0^\infty e^{-x}\frac{x^p}{\lfloor p}\left\{\int_0^\infty \frac{e^{-y}}{x+y}\cdot\frac{y^{p+q-1}}{\lfloor p+q-1}\,dy\right\}^n dx$$

$$= \frac{1}{q^n + p\left(\frac{\lfloor q-1}{\lfloor q-n} - q^{n-1}\right)}$$

$$\frac{1}{\sin^2\theta} - 8\left(\frac{1\cos 2\theta}{e^{2y}-1} + \frac{2\cos 4\theta}{e^{4y}-1} + \frac{3\cos 6\theta}{e^{6y}-1} + \&c\right)$$

$$= \frac{z^2}{\sin^2\phi} - z^2\left(\frac{1+x}{3}\right) + \frac{1}{3}\left\{1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)\right\}$$

$$\frac{\cos\theta}{\sin^3\theta} - 8\left(\frac{1^2\sin 2\theta}{e^{2y}-1} + \frac{2^2\sin 4\theta}{e^{4y}-1} + \&c\right)$$

$$= z^3\frac{\cos\phi}{\sin^3\phi}\sqrt{1 - x\sin^2\phi}$$

when $\theta z = \int_0^\phi \frac{d\phi}{\sqrt{1 - x\sin^2\phi}}$.

$$\sec\theta + 4\left\{\frac{\cos\theta}{e^{y}-1} - \frac{\cos 3\theta}{e^{3y}-1} + \frac{\cos 5\theta}{e^{5y}-1} - \&c\right\}$$

$$= z\sec\phi\sqrt{1 - x\sin^2\phi}.$$

$$\frac{\sin\theta}{\cosh\frac{y}{2}} - \frac{\sin 3\theta}{\cosh\frac{3y}{2}} + \frac{\sin 5\theta}{\cosh\frac{5y}{2}} - \&c =$$

$$= \frac{z}{2}\sqrt{x(1-x)}\cdot\frac{\sin\phi}{\sqrt{1 - x\sin^2\phi}}.$$

If n is odd $\phi(n-2)+\phi(n) =$

$$x\left\{\left(\frac{\sin x}{1^{n-2}} - \frac{\sin 3x}{3^{n-2}} + \frac{\sin 5x}{5^{n-2}} - \&c\right) - \left(\frac{\sin x}{1^n} - \frac{\sin 3x}{3^n} + \frac{\sin 5x}{5^n} - \&c\right)\right\}$$

$$+ n\left\{\left(\frac{\cos x}{1^{n-1}} - \frac{\cos 3x}{3^{n-1}} + \frac{\cos 5x}{5^{n-1}} - \&c\right) - \left(\frac{\cos x}{1^{n+1}} - \frac{\cos 3x}{3^{n+1}} + \frac{\cos 5x}{5^{n+1}} - \&c\right)\right\}$$

4. i. $\frac{1}{2}\sin 2x - \frac{1+\frac{1}{2}}{3}\sin 3x + \frac{1+\frac{1}{2}+\frac{1}{3}}{4}\sin 4x - \&c$

$= \frac{x}{2}\left(\frac{\cos x}{1} - \frac{\cos 2x}{2} + \frac{\cos 3x}{3} - \&c\right)$

ii. $\cos 2x - (1+\frac{1}{3})\cos 4x + (1+\frac{1}{3}+\frac{1}{5})\cos 6x - \&c$

$= \frac{\pi}{4}(\cos x - \cos 3x + \cos 5x - \&c)$

Integrate both sides in the above results n times.

5. i. $\sin a\theta + \frac{n}{\underline{1}}\sin(a+2)\theta + \frac{n(n-1)}{\underline{2}}\sin(a+4)\theta + \&c$

$= (2\cos\theta)^n \sin(a+n)\theta.$

ii. $\cos a\theta + \frac{n}{\underline{1}}\cos(a+2)\theta + \frac{n(n-1)}{\underline{2}}\cos(a+4)\theta + \&c$

$= (2\cos\theta)^n \cos(a+n)\theta$

Sol. find $e^{a\theta c} + \frac{n}{\underline{1}}e^{(a+2)\theta c} + \frac{n(n-1)}{\underline{2}}e^{(a+4)\theta c}$
and apply the De Moivre's theorem.

6. i. $\sin x - \frac{1}{2}\sin 3x + \frac{1.3}{2.4}\sin 5x - \&c =$

$\frac{1}{2}\sin 2x - \frac{1.3}{2.4}\sin 4x + \frac{1.3.5}{2.4.6}\sin 6x - \&c = \frac{\sin\frac{x}{2}}{\sqrt{2\cos x}}$

ii. $\cos x - \frac{1}{2}\cos 3x + \frac{1.3}{2.4}\cos 5x - \&c$

$= 1 - \frac{1}{2}\cos 2x + \frac{1.3}{2.4}\cos 4x - \frac{1.3.5}{2.4.6}\cos 6x + \&c = \frac{\cos\frac{x}{2}}{\sqrt{2\cos x}}$

iii. $\sin 2x - \frac{1}{2}\sin 4x + \frac{1.3}{2.4}\sin 6x - \&c = \frac{\sin\frac{3x}{2}}{\sqrt{2\cos x}}$

iv. $\cos 2x - \frac{1}{2}\cos 4x + \frac{1.3}{2.4}\cos 6x - \&c = \frac{\cos\frac{3x}{2}}{\sqrt{2\cos x}}$

v. $\frac{\sin 2x}{2} - \frac{1}{2}\frac{\sin 4x}{4} + \frac{1.3}{2.4}\frac{\sin 6x}{6} - \&c$ $\sin\frac{x}{2}\sqrt{2\cos x}$

$$\frac{n}{n+}\;\frac{x}{n+1+}\;\frac{x}{n+2+}\cdots+\frac{x}{x+n}.$$

$$\div\left\{1+\frac{nx}{\lfloor 1\; n(x+n)}+\frac{(n-1)(n-2)x^2}{\lfloor 2\; n(n+1)(n+n)(n+n-1)}+\frac{(n-2)(n-3)(n-4)x^3}{\lfloor 3\; n(n+1)(n+2)(n+n)(n+n-1)(n+n-2)}+\&c\right\}$$

$$\div\left\{1+\frac{(n-1)x}{\lfloor 1\,(n+1)(n+n)}+\frac{(n-2)(n-3)x^2}{\lfloor 2\,(n+1)(n+2)(n+n)(n+n-1)}+\&c\right\}$$

no. of terms being limited.

$$\frac{1}{1+}\;\frac{x}{1+}\;\frac{x}{1+}\;\frac{2x}{1+}\;\frac{2x}{1+}\cdots+\frac{(n-1)x}{1+}\;\frac{nx}{1}.$$

$$\div\; D=1+\frac{n^2}{\lfloor 1}x+\frac{n^2(n-1)^2}{\lfloor 2}x^2+\frac{n^2(n-1)^2(n-2)^2}{\lfloor 3}x^3+\&c$$

$$\frac{1}{1+}\;\frac{x}{1+}\;\frac{x}{1+}\;\frac{2x}{1+}\cdots+\frac{(n-1)x}{1+}\;\frac{(n-1)x}{1}.$$

$$D=1+\frac{n^2}{\lfloor 1}\left(1-\frac{1}{n}\right)x+\frac{n^2(n-1)^2}{\lfloor 2}\left(1-\frac{2}{n}\right)x^2+\frac{n^2(n-1)^2(n-2)^2}{\lfloor 3}\left(1-\frac{3}{n}\right)x^3+\&c$$

If $\phi(a,n)=1+ax\cdot\frac{1-x^n}{1-x}+a^2x^4\cdot\frac{(1-x^{n-1})(1-x^{n-2})}{(1-x)(1-x^2)}+a^3x^9\cdot\frac{(1-x^{n-2})(1-x^{n-3})(1-x^{n-4})}{(1-x)(1-x^2)(1-x^3)}+\&c$, then

$$\frac{\phi(ax,n-1)}{\phi(a,n)}=\frac{1}{1+}\;\frac{ax}{1+}\;\frac{ax^2}{1+}\;\frac{ax^3}{1+}\cdots+\frac{ax^n}{1}.$$

If $\int_0^{\alpha i}\frac{d\phi}{\sqrt{1-x\sin^2\phi}}=i\int_0^{\beta}\frac{d\phi}{\sqrt{1-(1-x)\sin^2\phi}}$

then $\alpha=\log\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right)$.

vi. $\frac{\cos 2x}{2} - \frac{1}{2}\cdot\frac{\cos 4x}{4} + \frac{1.3}{2.4}\cdot\frac{\cos 6x}{6} - \&c = $ ~~$\cos\frac{x}{2}$~~ $\sqrt{2\cos x} - 1$.

vii. $\frac{\sin x}{1} - \frac{1}{2}\cdot\frac{\sin 3x}{3} + \frac{1.3}{2.4}\cdot\frac{\sin 5x}{5} - \&c = \sin^{-1}\left(\sqrt{2}\sin\frac{x}{2}\right)$.

viii. $\frac{\cos x}{1} - \frac{1}{2}\cdot\frac{\cos 3x}{3} + \frac{1.3}{2.4}\cdot\frac{\cos 5x}{5} - \&c = \log_e\left(\sqrt{\cos x} + \sqrt{2}\cos\frac{x}{2}\right)$

ix. $\frac{\sin 2x}{2^2} - \frac{1}{2}\cdot\frac{\sin 4x}{4^2} + \frac{1.3}{2.4}\cdot\frac{\sin 6x}{6^2} - \&c = \sin\frac{x}{2}\sqrt{2\cos x}$

$$+ \sin^{-1}\left(\sqrt{2}\sin\frac{x}{2}\right) - x.$$

x. $\frac{\cos 2x}{2^2} - \frac{1}{2}\cdot\frac{\cos 4x}{4^2} + \frac{1.3}{2.4}\cdot\frac{\cos 6x}{6^2} - \&c = \cos\frac{x}{2}\sqrt{2\cos x}$

$$- \log_e\left(\sqrt{\cos x} + \sqrt{2}\cos\frac{x}{2}\right) - 1 + \log_e 2.$$

Similarly we can form similar identities for the II part also. From the results of the I part the following theorem is obtained.

7. Let $F(n) = \left\{\frac{\sin x}{1^n} - \frac{1}{2}\cdot\frac{\sin 3x}{3^n} + \frac{1.3}{2.4}\cdot\frac{\sin 5x}{5^n} - \&c\right\}$

$$- \cos\pi n\left\{\left(\frac{\sin 2x}{2^n} - \frac{1}{2}\cdot\frac{\sin 4x}{4^n} + \frac{1.3}{2.4}\cdot\frac{\sin 6x}{6^n} - \&c\right) - \left(\frac{\sin 2x}{2^{n+1}} - \frac{1}{2}\cdot\frac{\sin 4x}{4^{n+1}} + \frac{1.3}{2.4}\cdot\frac{\sin 6x}{6^{n+1}} - \&c\right)\right\}$$ and

$\psi(n) = \left\{\frac{\cos x}{1^n} - \frac{1}{2}\cdot\frac{\cos 3x}{3^n} + \frac{1.3}{2.4}\cdot\frac{\cos 5x}{5^n} - \&c\right\}$

$$+ \cos\pi n\left\{\left(\frac{\cos 2x}{2^n} - \frac{1}{2}\cdot\frac{\cos 4x}{4^n} + \frac{1.3}{2.4}\cdot\frac{\cos 6x}{6^n} - \&c\right) - \left(\frac{\cos 2x}{2^{n+1}} - \frac{1}{2}\cdot\frac{\cos 4x}{4^{n+1}} + \frac{1.3}{2.4}\cdot\frac{\cos 6x}{6^{n+1}} - \&c\right)\right\}$$, then

If n is odd,

i. $\frac{F(n)}{2}\sin\frac{\pi n}{2} = \frac{x^n}{\lfloor\underline{n}}S_0\phi(0) - \frac{x^{n-2}}{\lfloor\underline{n-2}}\left\{S_0\phi(2) + \frac{S_2}{2^2}\phi(0)\right\}$

$$+ \frac{x^{n-4}}{\lfloor\underline{n-4}}\left\{S_0\phi(4) + \frac{S_2}{2^2}\phi(2) + \frac{S_4}{2^4}\phi(0)\right\} - \&c$$

$$= \frac{A_{n-1}}{\lfloor\underline{n-1}}\phi(0) - \frac{A_{n-3}}{\lfloor\underline{n-3}}\phi(2) + \frac{A_{n-5}}{\lfloor\underline{n-5}}\phi(4) - \&c \qquad \text{I}.$$

ii. $\frac{F(n+1)}{2}\sin\pi n = \frac{x^n}{\lfloor\underline{n}}S_0\phi(1) - \frac{x^{n-2}}{\lfloor\underline{n-2}}\left\{S_0\phi(3) + \frac{S_2}{2^2}\phi(1)\right\}$

$$\frac{(1-e^{-x})}{1^2} + \frac{(1-e^{-x})^2}{2^2} + \frac{(1-e^{-x})^3}{3^2} + \&c$$

$$= x - \frac{x^2}{4} + B_2 \frac{x^3}{\lfloor 3} - B_4 \frac{x^5}{\lfloor 5} + B_6 \frac{x^7}{\lfloor 7} - \&c$$

$$\begin{cases} S = \dfrac{\sin\frac{\theta}{2}}{\sinh\frac{y}{2}} + \dfrac{\sin\frac{3\theta}{2}}{\sinh\frac{3y}{2}} + \dfrac{\sin\frac{5\theta}{2}}{\sinh\frac{5y}{2}} + \&c \\ C = \dfrac{\cos\frac{\theta}{2}}{\cosh\frac{y}{2}} + \dfrac{\cos\frac{3\theta}{2}}{\cosh\frac{3y}{2}} + \dfrac{\cos\frac{5\theta}{2}}{\cosh\frac{5y}{2}} + \&c \\ C_1 = \dfrac{1}{2} + \dfrac{\cos\theta}{\cosh y} + \dfrac{\cos 2\theta}{\cosh 2y} + \dfrac{\cos 3\theta}{\cosh 3y} + \&c \end{cases}$$

$$CS = \frac{\sin\theta}{\cosh y} + \frac{2\sin 2\theta}{\cosh 2y} + \frac{3\sin 3\theta}{\cosh 3y} + \&c$$

$$\therefore CS + \frac{dC_1}{d\theta} = 0 \quad \therefore C_1 S + \frac{dC}{d\theta} = 0 \ \& \ CC_1 = \frac{dS}{d\theta}.$$

$$\because C^2 + S^2 = x\frac{z^2}{4} \ \& \ C_1^2 + S^2 = \frac{z^2}{4}.$$

$$\therefore \text{Let } C = \sqrt{x}\cdot\frac{z}{2}\cos\phi \ \& \ S = \sqrt{x}\cdot\frac{z}{2}\sin\phi.$$

$$\therefore C_1 = \frac{z}{2}\sqrt{1 - x\sin^2\phi}$$

$$\therefore \frac{z}{2}\cos\phi\sqrt{1 - x\sin^2\phi} = \frac{d\sin\phi}{d\theta} = \cos\phi\frac{d\phi}{d\theta}$$

$$\therefore \theta = \frac{2}{z}\int_0^{\phi}\frac{d\phi}{\sqrt{1 - x\sin^2\phi}}.$$

$\sin\phi$ to $i\tan\phi$
$\cos\phi$ to $\sec\phi$
ϕ to $i\log\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$
θ to $i\theta\frac{z}{z'}$
y to y'
x to $1-x$
z to z'

$+ \frac{x^{n-4}}{\lfloor n-4} \left\{ S_0 \phi(5) + \frac{S_2}{\lfloor 2} \phi(3) + \frac{S_4}{\lfloor 4} \phi(1) \right\} - \&c$

$= \frac{A_{n-1}}{\lfloor n-1} \phi(1) - \frac{A_{n-3}}{\lfloor n-3} \phi(3) + \frac{A_{n-5}}{\lfloor n-5} \phi(5) - \&c$ II.

where $S_n = \frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \&c$

$\frac{\pi}{2} \phi(n) = \frac{1}{1^{n+1}} + \frac{1}{2} \cdot \frac{1}{3^{n+1}} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5^{n+1}} + \&c$

and $\frac{2}{\pi} A_n = \left(\frac{\pi}{2}\right)^n + \left(3\frac{\pi}{2}\right)^n + \left(5\frac{\pi}{2}\right)^n + \cdots + \left(x\frac{\pi}{2}\right)^n$

If n is even $\frac{\psi(n)}{2} \cos\frac{\pi n}{2} = I$ & $\frac{\psi(n+1)}{2} \cos\frac{\pi n}{2} = II$

8. If $\phi(x) = \frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \&c$, then

i. $\phi(x-1) + \phi\left(\frac{1}{x} - 1\right) = \frac{1}{2}(\log_e x)^2$

ii. $\phi(x) + \phi\left(\frac{1}{x}\right) = \frac{1}{2}(\log_e x)^2 + \frac{\pi^2}{6}$

iii. $\phi(-x) + \phi(x-1) = \log_e x \log_e(1-x) - \frac{\pi^2}{6}$

iv. $\phi(x) + \phi(-x) = \frac{1}{2}\phi(-x^2)$

v. $\phi(x) - \phi(-x) = \psi(x) = 2\left(\frac{x}{1^2} + \frac{x^3}{3^2} + \frac{x^5}{5^2} + \&c\right)$

$\psi(x) + \psi\left(\frac{1-x}{1+x}\right) = \frac{\pi^2}{4} + \log_e x \log_e \frac{1+x}{1-x}$

Ex. 1. $\frac{1}{1^2} \cdot \frac{1}{2} + \frac{1}{2^2} \cdot \frac{1}{2^2} + \frac{1}{3^2} \cdot \frac{1}{2^3} + \frac{1}{4^2} \cdot \frac{1}{2^4} + \&c = \frac{\pi^2}{12} - \frac{1}{2}(\log_e 2)^2$

2. $\frac{1}{1^2}\left(\frac{\sqrt5-1}{2}\right) + \frac{1}{2^2}\left(\frac{\sqrt5-1}{2}\right)^2 + \frac{1}{3^2}\left(\frac{\sqrt5-1}{2}\right)^3 + \&c = \frac{\pi^2}{10} - \left(\log_e \frac{\sqrt5+1}{2}\right)^2$

3. $\frac{1}{1^2}\left(\frac{3-\sqrt5}{2}\right) + \frac{1}{2^2}\left(\frac{3-\sqrt5}{2}\right)^2 + \frac{1}{3^2}\left(\frac{3-\sqrt5}{2}\right)^3 + \&c = \frac{\pi^2}{15} - \left(\log_e \frac{\sqrt5+1}{2}\right)^2$

4. $\frac{\sqrt2-1}{1^2} + \frac{(\sqrt2-1)^3}{3^2} + \frac{(\sqrt2-1)^5}{5^2} + \&c = \frac{\pi^2}{16} - \frac{1}{4}\left\{\log_e(1+\sqrt2)\right\}^2$

5. $\frac{1}{1^2}\left(\frac{\sqrt5-1}{2}\right) + \frac{1}{3^2}\left(\frac{\sqrt5-1}{2}\right)^3 + \frac{1}{5^2}\left(\frac{\sqrt5-1}{2}\right)^5 + \&c = \frac{\pi^2}{12} - \frac{3}{4}\left(\log_e \frac{\sqrt5+1}{2}\right)^2$

6. $\frac{\sqrt5-2}{1^2} + \frac{(\sqrt5-2)^3}{3^2} + \frac{(\sqrt5-2)^5}{5^2} + \&c = \frac{\pi^2}{24} - \frac{3}{4}\left(\log_e \frac{\sqrt5+1}{2}\right)^2$

7. If $\phi(x) = \frac{x}{1^3} - \frac{x^2}{2^3} + \frac{x^3}{3^3} - \frac{x^4}{4^3} + \&c$, then

If $F(\alpha, \beta) = \frac{\alpha}{n+} \frac{(\beta)^2}{n+} \frac{(2\alpha)^2}{n+} \frac{(3\beta)^2}{n+}$ &c

~~then $F\alpha, \beta$~~ & A, G the ~~A.M.~~ & G.M. between α & β

then $F(A, G)$ is the A.M between $F(\alpha, \beta)$ & $F(\beta, \alpha)$

If $a_1 + a_2 = b_1 + b_2 = c_1 + c_2 =$ &c $= p$

$a_2 a_3 = b_2 b_3 = c_2 c_3 = d_2 d_3$ &c $= q$

$a_3 + a_4 = b_3 + b_4 = c_3 + c_4 =$ &c $= r$

$a_4 a_5 = b_4 b_5 = c_4 c_5 =$ &c $= s$ &c &c

then $\Sigma \frac{a}{1+} \frac{a_1}{1+} \frac{a_2}{1+} \frac{a_3}{1+}$ &c

$$= \Sigma a - \cfrac{\Sigma a a_1}{1 + p - \cfrac{q}{1 + r - \cfrac{s}{1 + t - \cfrac{u}{1 + v - \&c}}}}$$

If $F(\alpha, \beta) = \tan^{-1} \frac{\alpha}{x+} \frac{\beta^2 + k^2}{x+} \frac{\alpha^2 + (2k)^2}{x+} \frac{\beta^2 + (3k)^2}{x + \&c}$

and A the A.M between α & β, then

$F(A, A)$ is the A.M between $F(\alpha, \beta)$ & $F(\beta, \alpha)$.

i. $\phi(x-1) + \phi(\frac{1}{x}-1) + \phi(-x) = \frac{1}{2}(\log_e x)^2 \log_e(1-x) - \frac{1}{6}(\log_e x)^3 - \frac{\pi^2}{6}\log_e x - S_3$.

ii. $\phi(x) - \phi(\frac{1}{x}) = \frac{1}{6}(\log_e x)^3 + \frac{\pi^2}{6}\log_e x$.

iii. $\phi(x) + \phi(-x) = \frac{1}{4}\phi(-x^2)$.

Ex 1. $\frac{1}{1^3}\cdot\frac{1}{2} + \frac{1}{2^3}\cdot\frac{1}{2^2} + \frac{1}{3^3}\cdot\frac{1}{2^3} + \frac{1}{4^3}\cdot\frac{1}{2^4} + \&c = \frac{1}{6}(\log_e 2)^3 - \frac{\pi^2}{12}\log_e 2 + (\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c)$

2. $\frac{1}{1^3}\cdot\frac{3-\sqrt{5}}{2} + \frac{1}{2^3}\cdot\left(\frac{3-\sqrt{5}}{2}\right)^2 + \frac{1}{3^3}\cdot\left(\frac{3-\sqrt{5}}{2}\right)^3 + \&c$
$= \frac{2}{3}\left(\log_e \frac{\sqrt{5}+1}{2}\right)^3 - \frac{2\pi^2}{15}\log_e\frac{\sqrt{5}+1}{2} + S_3$

10. If $\phi(x) = x + (1+\frac{1}{3})\frac{x^3}{3} + (1+\frac{1}{3}+\frac{1}{5})\frac{x^5}{5} + \&c$

then $\phi\left(\frac{x}{2-x}\right) = \frac{1}{8}\{\log_e(1-x)\}^2 + \frac{1}{2}\left(\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \&c\right)$

Ex 1. $\frac{1}{3} + \frac{1+\frac{1}{3}}{3}\cdot\frac{1}{3^3} + \frac{1+\frac{1}{3}+\frac{1}{5}}{5}\cdot\frac{1}{3^5} + \&c = \frac{\pi^2}{24} - \frac{1}{8}(\log_e 2)^2$

2. $\frac{1}{5} + \frac{1+\frac{1}{3}}{3}\cdot\frac{1}{5^3} + \frac{1+\frac{1}{3}+\frac{1}{5}}{5}\cdot\frac{1}{5^5} + \&c = \frac{\pi^2}{20\sqrt{5}}$
$- \frac{3}{8\sqrt{5}}\left(\log_e\frac{\sqrt{5}+1}{2}\right)^2$

3. $(\sqrt{5}-2) + \frac{1+\frac{1}{3}}{3}(\sqrt{5}-2)^3 + \frac{1+\frac{1}{3}+\frac{1}{5}}{5}(\sqrt{5}-2)^5 + \&c$
$= \frac{\pi^2}{30} - \frac{3}{4}\left(\log_e\frac{\sqrt{5}+1}{2}\right)^2$

11. If $\phi(x) = \frac{x^2}{2^2} + (1+\frac{1}{2})\frac{x^3}{3^2} + (1+\frac{1}{2}+\frac{1}{3})\frac{x^4}{4^2} + \&c$, then

i. $\phi(1-x) = \frac{1}{2}\log_e(1-x)(\log_e x)^2 + \log_e x\left(\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \&c\right)$
$- \left(\frac{x}{1^3} + \frac{x^2}{2^3} + \frac{x^3}{3^3} + \&c\right) + S_3$

ii. $\phi(1-x) - \phi(1-\frac{1}{x}) = \frac{1}{6}(\log_e x)^3$

iii. $\phi(1-x) = \frac{1}{2}\log_e(1-x)(\log_e x)^2 - \frac{1}{3}(\log_e x)^3 - \log_e x\left(\frac{1}{1^2 x} + \frac{1}{2^2 x^2} + \&c\right)$
$- \left(\frac{1}{1^3 x} + \frac{1}{2^3 x^2} + \frac{1}{3^3 x^3} + \&c\right) + S_3$

$$\int_0^\infty e^{-x}\left(1+\frac{x}{n}\right)^n dx = 1+\frac{n}{1+}\frac{1(n-1)}{3+}\frac{2(n-2)}{5+}\frac{3(n-3)}{7+\&c}$$

$$= 2+\frac{n-1}{2+}\frac{1(n-2)}{4+}\frac{2(n-3)}{6+}\frac{3(n-4)}{8+\&c}$$

$$= \frac{e^n \lfloor n}{n^n} - \frac{2n}{2+}\frac{3n}{3+}\frac{4n}{4+}\frac{5n}{5+}$$

$$\frac{\operatorname{cosech}\frac{y}{2}}{1+n^2} - \frac{\operatorname{cosech}\frac{3y}{2}}{1+(3n)^2} + \frac{\operatorname{cosech}\frac{5y}{2}}{1+(5n)^2} - \&c$$

$$= \frac{\frac{z}{2}\sqrt{x}}{1+}\frac{(1-x)(nz)^2}{1-}\frac{x(2nz)^2}{1+}\frac{(1-x)(3nz)^2}{1-\&c}$$

$$\int_0^\infty e^{-n\int_0^\phi \frac{d\theta}{\sqrt{1-x\sin^2\theta}}} \cdot \frac{\cos\phi}{1-x\sin^2\phi}\,d\phi$$

$$= \frac{1}{n+}\frac{1-x}{n-}\frac{4x}{n+}\frac{9(1-x)}{n-}\frac{16x}{n+\&c}$$

If $1+\sqrt{1-\alpha} = \frac{2}{1+\sqrt{\beta}} = \gamma$

then $\frac{\sqrt{\beta}}{n+}\frac{1^2}{n+}\frac{2^2\beta}{n+}\frac{3^2}{n+}\frac{4^2\beta}{n+}\frac{5^2}{n+}\&c$

$+\frac{1}{n+}\frac{1^2\beta}{n+}\frac{2^2}{n+}\frac{3^2\beta}{n+}\frac{4^2}{n+}\&c$

$= \frac{2}{n\gamma+}\frac{1^2\alpha}{n\gamma+}\frac{2^2}{n\gamma+}\frac{3^2\alpha}{n\gamma+}\frac{4^2}{n\gamma+}\&c$

iv. $\phi(-x) + \phi\left(-\frac{1}{x}\right) = -\frac{1}{6}(\log_e x)^3 + \log_e x\left(\frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c\right)$

$$-\left(\frac{x}{1^3} - \frac{x^2}{2^3} + \frac{x^3}{3^3} - \&c\right) + S_3$$

12. If $\phi(x) = \frac{x^2}{2^3} - \left(1+\frac{1}{2}\right)\frac{x^3}{3^3} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^4}{4^3} - \&c$, then

i. $\phi(x-1) - \phi\left(\frac{1}{x}-1\right) = -\frac{1}{24}(\log_e x)^4 + \frac{1}{6}(\log_e x)^3 \log_e(1-x)$

$$+ \log_e x\left(\frac{x}{1^3} + \frac{x^2}{2^3} + \frac{x^3}{3^3} + \&c\right) - 2\left(\frac{x}{1^4} + \frac{x^2}{2^4} + \frac{x^3}{3^4} + \&c\right)$$

$$+ S_3 \log_e x + \frac{\pi^4}{45}$$

ii. $\phi(x) - \phi\left(\frac{1}{x}\right) = -\frac{1}{24}(\log_e x)^4 + \log_e x\left(\frac{x}{1^3} - \frac{x^2}{2^3} + \frac{x^3}{3^3} - \&c\right)$

$$-2\left(\frac{x}{1^4} - \frac{x^2}{2^4} + \frac{x^3}{3^4} - \&c\right) + S_3 \log_e x + \frac{7\pi^4}{360}.$$

13. If $\phi(x) = \frac{x^2}{2^2} + \left(1+\frac{1}{3}\right)\frac{x^4}{4^2} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{x^6}{6^2} + \&c$ and

$\psi(x) = \frac{x^2}{2^3} + \left(1+\frac{1}{3}\right)\frac{x^4}{4^3} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{x^6}{6^3} + \&c$ then

i. $\phi\left(\frac{1-x}{1+x}\right) = \frac{1}{8}(\log_e x)^2 \log_e\frac{1-x}{1+x} + \frac{1}{2}\log_e x\left(\frac{x}{1^2} + \frac{x^3}{3^2} + \frac{x^5}{5^2} + \&c\right)$

$$+ \frac{1}{2}\left(\frac{1-x}{1^3} + \frac{1-x^3}{3^3} + \frac{1-x^5}{5^3} + \&c\right)$$

ii. $\psi(x) + \psi\left(\frac{1-x}{1+x}\right) = \phi(x)\log_e x + \phi\left(\frac{1-x}{1+x}\right)\log_e\frac{1-x}{1+x}$

$$-\frac{1}{16}(\log_e x)^2\left(\log_e\frac{1-x}{1+x}\right)^2$$

$$+\frac{\pi}{4}\left(\frac{1}{1^3} - \frac{1}{5^3} + \frac{1}{9^3} - \&c\right) - \frac{\pi}{3\sqrt{3}}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$$

14. If $\phi(x) = x + \left(1+\frac{1}{2}\right)\frac{x^3}{3} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^5}{5} + \&c$ then

$\phi\left(\frac{1-x}{1+x}\right) = -(1-\log_e 2)\log_e x + \frac{1+x}{1-x}\log_e\frac{4x}{(1+x)^2} + \frac{1}{4}(\log_e x)^2$

$$-\left(\frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c\right) + \frac{\pi^2}{12}$$

Ex. 1. $\frac{1}{2} + \frac{1+\frac{1}{2}}{2^2}\cdot\frac{1}{2^2} + \frac{1+\frac{1}{2}+\frac{1}{3}}{3^2}\cdot\frac{1}{2^3} + \&c = S_3 - \frac{\pi^2}{12}\log_e 2$

2. $\frac{1}{1^2} + \frac{1+\frac{1}{2}+\frac{1}{3}}{3^2} + \frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}}{5^2} + \&c = \frac{3}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \right.$

If $I(n)$ be the nearest integer to

$$\frac{1}{4n}\left\{\cosh \pi\sqrt{n} - \frac{\sinh \pi\sqrt{n}}{\pi\sqrt{n}}\right\}$$

then $I(0) + xI(1) + x^2 I(2) + x^3 I(3) + \&c$

$$= \frac{1}{1 - 2x + 2x^4 - 2x^9 + 2x^{16} - \&c}.$$

If $\displaystyle\int_0^\infty \frac{e^{-m^2x^2}}{1+x^2}\,dx = \phi(m)$, & if $m \not< n$.

then $$\int_0^\infty \frac{e^{-m^2x^2}}{1+x^2}\cos 2mnx\,dx = \frac{1}{2}e^{-n^2}\left\{\phi(m+n) + \phi(m-n)\right\}$$

$$\int_0^\infty e^{-x}\left(1+\frac{x}{n}\right)^n dx = \frac{e^n \lfloor\underline{n}}{2n^n} + \frac{2}{3} - \frac{4}{135n} + \frac{8}{27.105n^2} + \frac{16}{105.81n^3} {}^{*}$$

$$(m-n-1)\int_0^\infty \frac{\left(1+\frac{x}{n}\right)^n}{\left(1+\frac{x}{m}\right)^m}dx = \frac{m}{2}\cdot\frac{m^m\lfloor\underline{n}}{n^n\lfloor\underline{m}}\cdot\frac{\lfloor\underline{m-n}}{(m-n)^{m-n}} + \frac{2}{3}(m+n) - \frac{4(m+n)(m-2n)(m-\frac{n}{2})}{135mn(m-n)} + \frac{8(m^3+n^3)(m-2n)(m-\frac{n}{2})}{27.105m^2n^2(m-n)^2} + \frac{16(m^3+n^3)}{105.81m^3n^3} - {}^{*}\frac{32.281}{3^8.5^2.7.11n^4} - \&c \times \frac{(m^2-mn+n^2)(m-2n)\times(m-\frac{n}{2})}{(m-n)^3},$$

3. $\frac{1}{1^2} + \frac{1+\frac{1}{3}}{2^2} + \frac{1+\frac{1}{3}+\frac{1}{5}}{3^2} + \&c = 2\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$

4. $(\sqrt{5}-2) + \frac{1+\frac{1}{2}}{3}(\sqrt{5}-2)^3 + \frac{1+\frac{1}{2}+\frac{1}{3}}{5}(\sqrt{5}-2)^5 + \&c$

$= \frac{\pi^2}{60} + \frac{3}{4}\left(\log_e \frac{\sqrt{5}+1}{2}\right)^2 + (\sqrt{5}+2)\log_e 4 + (3\sqrt{5}+5+\log_e 2)\log_e \frac{\sqrt{5}-1}{2}$

15. $S_{n+1}\cos\frac{\pi n}{2}\,\lfloor n = \int \frac{x^n}{2}\cot\frac{x}{2}\,dx +$

$x^n\left(\frac{\cos x}{1} + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \&c\right)$

$- n x^{n-1}\left(\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \&c\right)$

$- n(n-1)x^{n-2}\left(\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \&c\right)$

$+ n(n-1)(n-2)x^{n-3}\left(\frac{\sin x}{1^3} + \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} + \&c\right)$

$+ \&c$ where $S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c$ & $S_1 = \log_e 2$

Sol. $\sin x + \sin 2x + \sin 3x + \&c = \frac{1}{2}\cot\frac{x}{2}$

$\therefore \int x^n(\sin x + \sin 2x + \&c)\,dx = \int \frac{x^n}{2}\cot\frac{x}{2}\,dx$

and apply $\int uv\,dx = uv_1 - u'v_2 + u''v_3$ &c.

16. $S_{n+1}\cos\frac{\pi n}{2}\,\lfloor n = \int \frac{x^n}{2\sin x}\,dx$

$+ x^n\left(\frac{\cos x}{1} + \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \&c\right)$

$- n x^{n-1}\left(\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \&c\right)$

$- n(n-1)x^{n-2}\left(\frac{\cos x}{1^3} + \frac{\cos 3x}{3^3} + \frac{\cos 5x}{5^3} + \&c\right) + \&c$

Sol. $\sin x + \sin 3x + \sin 5x + \&c = \frac{1}{2}\operatorname{cosec} x$.

17. If $\int x^n \cot x\,dx = f_n(x)$, then

$2^n f_n\left(\frac{\pi}{2}-x\right) = \pi^n\{f_0(2x) - f_0(x)\} - \frac{n}{\lfloor 1}\pi^{n-1}\{f_1(2x) - 2f_1(x)\}$

$+ \frac{n(n-1)}{\lfloor 2}\pi^{n-2}\{f_2(2x) - 2^2 f_2(x)\} - \frac{n(n-1)(n-2)}{\lfloor 3}\pi^{n-3}$ [illegible]

$\{f_3(2x) - 2^3 f_3(x)\}$ [illegible]

$$\frac{\operatorname{Sech}\frac{y}{2}}{1+n^2} + \frac{\operatorname{Sech}\frac{3y}{2}}{1+(3n)^2} + \frac{\operatorname{Sech}\frac{5y}{2}}{1+(5n)^2} + \&c$$

$$= \frac{1}{2} \cdot \cfrac{z\sqrt{x}}{1+\cfrac{(nz)^2}{1+\cfrac{(2nz)^2 x}{1+\cfrac{(3nz)^2}{1+\cfrac{(4nz)^2 x}{1+\&c}}}}}$$

$$\int_0^\infty e^{-n\int_0^\phi \frac{d\theta}{\sqrt{1-x\sin^2\theta}}} \cdot \frac{\cos\phi}{\sqrt{1-x\sin^2\phi}}\, d\phi = \cfrac{1}{n+\cfrac{1}{n+\cfrac{4x}{n+\cfrac{9}{n+\cfrac{16x}{n+\&c}}}}}$$

$$\frac{\operatorname{Sech}\frac{y}{2}}{1+n^2} - \frac{3\operatorname{Sech}\frac{3y}{2}}{1+(3n)^2} + \frac{5\operatorname{Sech}\frac{5y}{2}}{1+(5n)^2} - \&c$$

$$= \frac{1}{2} \cdot \cfrac{z^2\sqrt{x(1-x)}}{1+n^2z^2(1-2x)+\cfrac{2^2(2^2-1)\,x(1-x)\,n^4z^4}{1+(3nz)^2(1-2x)+\cfrac{4^2(4^2-1)\,x(1-x)\,n^4z^4}{1+(5nz)^2(1-2x)+\&c}}}$$

$$\frac{\operatorname{Cosech}\frac{y}{2}}{1+n^2} + \frac{3\operatorname{Cosech}\frac{3y}{2}}{1+(3n)^2} + \frac{5\operatorname{Cosech}\frac{5y}{2}}{1+(5n)^2} + \&c$$

$$= \frac{1}{2} \cdot \cfrac{z^2\sqrt{x}}{1+(nz)^2(1+x)-\cfrac{2^2(2^2-1)\,x\,n^4z^4}{1+(3nz)^2(1+x)-\cfrac{4^2(4^2-1)\,x\,n^4z^4}{1+(5nz)^2(1+x)-\&c}}}$$

Sol. $\tan x = \cot x - 2\cot 2x$ and

$$f_n\left(\frac{\pi}{2}-x\right) = \int\left(\frac{\pi}{2}-x\right)^n \cot\left(\frac{\pi}{2}-x\right) d\left(\frac{\pi}{2}-x\right) = -\int\left(\frac{\pi}{2}-x\right)^n \tan x\,dx$$

N. B. Let $\sin x = y$ & $\tan x = z$, then

$$\int x^n \cot x\,dx = \int \frac{x^n}{\sin x}\cos x\,dx = \int \frac{(\sin^{-1}y)^n}{y}\,dy \text{ and}$$

$$\int \frac{2x^n}{\sin 2x}\,dx = \int \frac{x^n}{\cos x \sin x}\,dx = \int \frac{x^n}{\tan x}\sec^2 x\,dx$$

$$= \int \frac{(\tan^{-1}z)^n}{z}\,dz$$ So we have to find $(\sin^{-1}y)^n$ & $(\tan^{-1}y)^n$

i. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \&c$

ii. $\frac{1}{2}(\tan^{-1}x)^2 = \frac{x^2}{2} - \left(1+\frac{1}{3}\right)\frac{x^4}{4} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{x^6}{6} - \&c$

iii. $\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \&c$

iv. $\frac{1}{2}(\sin^{-1}x)^2 = \frac{x^2}{2} + \frac{2}{3}\cdot\frac{x^4}{4} + \frac{2.4}{3.5}\cdot\frac{x^6}{6} + \frac{2.4.6}{3.5.7}\cdot\frac{x^8}{8} + \&c$

v. $\frac{1}{6}(\sin^{-1}x)^3 = \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5}\left(1+\frac{1}{3^2}\right) + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}\left(1+\frac{1}{3^2}+\frac{1}{5^2}\right)$
$+ \&c$

vi. $\frac{1}{24}(\sin^{-1}x)^4 = \frac{2}{3}\cdot\frac{x^4}{4}\cdot\frac{1}{2^2} + \frac{2.4}{3.5}\cdot\frac{x^6}{6}\left(\frac{1}{2^2}+\frac{1}{4^2}\right) + \&c$

vii. $\frac{d(\sin^{-1}x)^n}{dx} = n\frac{(\sin^{-1}x)^{n-1}}{\sqrt{1-x^2}}$ & $\frac{d(\tan^{-1}x)^n}{dx} = n\cdot\frac{(\tan^{-1}x)^{n-1}}{1+x^2}$

18. $\frac{\sin x}{1^2} + \frac{1}{2}\cdot\frac{\sin^3 x}{3^2} + \frac{1.3}{2.4}\cdot\frac{\sin^5 x}{5^2} + \frac{1.3.5}{2.4.6}\cdot\frac{\sin^7 x}{7^2} + \&c$

$= x\log_e 2\sin x + \frac{1}{2}\left(\frac{\sin 2x}{1^2} + \frac{\sin 4x}{2^2} + \frac{\sin 6x}{3^2} + \&c\right)$

Ex 1. $\frac{1}{1^2} + \frac{1}{2}\cdot\frac{1}{3^2} + \frac{1.3}{2.4}\cdot\frac{1}{5^2} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{7^2} + \&c = \frac{\pi}{2}\log_e 2$

2. $\frac{1}{1^2} + \frac{1}{2}\cdot\frac{1}{3^2}\cdot\frac{1}{2} + \frac{1.3}{2.4}\cdot\frac{1}{5^2}\cdot\frac{1}{2^2} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{7^2}\cdot\frac{1}{2^3} + \&c$

$= \frac{\pi}{4\sqrt{2}}\log_e 2 + \frac{1}{\sqrt{2}}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c\right)$

3. $\frac{1}{1^2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3^2}\cdot\frac{1}{2^3} + \frac{1.3}{2.4}\cdot\frac{1}{5^2}\cdot\frac{1}{2^5} + \&c$

$= \frac{3\sqrt{3}}{4}\left(\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} + \&c\right) - \frac{\pi^2}{6\sqrt{3}}$

If $\alpha\beta = 4\pi^2$ & $F(x) = \dfrac{\sqrt{5}+1}{2} + \cfrac{\sqrt[5]{x}}{1+\cfrac{x}{1+\cfrac{x^2}{1+\cfrac{x^3}{1+\&c}}}}$

then $F(e^{-\alpha})\, F(e^{-\beta}) = \dfrac{5+\sqrt{5}}{2}$.

$$f^2(-x^2,-x^3) - \sqrt[5]{x^2}\, f^2(-x,-x^4)$$
$$= f(-x,-x^2)\left\{f(-\sqrt[5]{x}, -\sqrt[5]{x^2}) + \sqrt[5]{x}\, f(-x^5,-x^{10})\right\}.$$

If $K = \dfrac{f(-x^{\frac{1}{5}}, -x^{\frac{2}{5}})}{\sqrt[5]{x}\, f(-x^5,-x^{10})}$, then

$$\frac{\sqrt{K^2+2K+5}-(K+1)}{2} = \cfrac{\sqrt[5]{x}}{1+\cfrac{x}{1+\cfrac{x^2}{1+\cfrac{x^3}{1+\&c}}}}$$

If $\displaystyle\int_0^\infty \phi(x) \cos 2nx\, dx = \frac{\sqrt{\pi}}{2}\psi(n)$

then $\displaystyle\int_0^\infty e^{-x^2}\phi(x)\, dx = \int_0^\infty e^{-x^2}\psi(x)\, dx$.

&c &c &c

~~$\displaystyle\int_0^\infty \frac{F(a+ix) - F(a-ix)}{\sinh}$~~

4. $\frac{1}{1^2} + \frac{1}{2}\cdot\frac{1}{3^2}\cdot\left(\frac{3}{4}\right) + \frac{1.3}{2.4}\cdot\frac{1}{5^2}\left(\frac{3}{4}\right)^2 + \frac{1.3.5}{2.4.6}\cdot\frac{1}{7^2}\left(\frac{3}{4}\right)^3 + \&c$

$= \frac{\pi}{3\sqrt{3}}\log_e 3 - \frac{2\pi^2}{27} + \left(\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} + \&c\right)$

19. $\frac{\tan x}{1^2} - \frac{\tan^3 x}{3^2} + \frac{\tan^5 x}{5^2} - \frac{\tan^7 x}{7^2} + \&c$

$= x\log_e \tan x + \left(\frac{\sin 2x}{1^2} + \frac{\sin 6x}{3^2} + \frac{\sin 10x}{5^2} + \&c\right)$

Ex. 1. $\frac{1}{1^2}\cdot\frac{1}{3} - \frac{1}{3^2}\cdot\frac{1}{3^2} + \frac{1}{5^2}\cdot\frac{1}{3^3} - \&c$

$= -\frac{\pi}{12\sqrt{3}}\log_e 3 - \frac{5}{54}\pi^2 + \frac{5}{4}\left(\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} + \&c\right)$

2. $\frac{\sqrt{2}-1}{1^2} - \frac{(\sqrt{2}-1)^3}{3^2} + \frac{(\sqrt{2}-1)^5}{5^2} - \frac{(\sqrt{2}-1)^7}{7^2} + \&c$

$= -\frac{\pi}{8}\log_e(1+\sqrt{2}) - \frac{\pi^2}{16} + \sqrt{2}\left(\frac{1}{1^2} - \frac{1}{5^2} + \frac{1}{9^2} - \frac{1}{13^2} + \&c\right)$

3. $\frac{2-\sqrt{3}}{1^2} - \frac{(2-\sqrt{3})^3}{3^2} + \frac{(2-\sqrt{3})^5}{5^2} - \&c = -\frac{\pi}{12}\log_e(2+\sqrt{3})$

$+ \frac{2}{3}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c\right)$

20. If x lies between 0 & $\frac{\pi}{4}$

$\frac{\cos x - \sin x}{1^2} + \frac{1}{2}\cdot\frac{\cos^3 x - \sin^3 x}{3^2} + \frac{1.3}{2.4}\cdot\frac{\cos^5 x - \sin^5 x}{5^2} + \&c$

$+ \frac{1}{2}\left\{\frac{\sin 2x}{1^2} + \frac{1}{2}\cdot\frac{\sin^3 2x}{3^2} + \frac{1.3}{2.4}\cdot\frac{\sin^5 2x}{5^2} + \&c\right\}$

$= \frac{\pi}{2}\log_e 2\cos x.$

Ex. If $\psi(x) = \frac{x}{1^2} + \frac{1}{2}\cdot\frac{x^3}{3^2} + \frac{1.3}{2.4}\cdot\frac{x^5}{5^2} + \&c$

shew that $\psi\left(\frac{3}{5}\right) - \frac{1}{2}\psi\left(\frac{24}{25}\right) = \frac{\pi}{2}\log_e 2 + 2\psi\left(\frac{1}{\sqrt{5}}\right) - 3\psi(\ldots$

21. $\frac{\cos x + \sin x}{1^2} + \frac{1}{2}\cdot\frac{\cos^3 x + \sin^3 x}{3^2} + \frac{1.3}{2.4}\cdot\frac{\cos^5 x + \sin^5 x}{5^2} + \&c$

$= \frac{\pi}{2}\log_e 2\cos x + \frac{\tan x}{1^2} - \frac{\tan^3 x}{3^2} + \frac{\tan^5 x}{5^2} - \&c$

Ex. $\frac{1}{1^2}\cdot\frac{2+1}{5} + \frac{1}{2}\cdot\frac{1}{3^2}\cdot\frac{2^3+1}{5^2} + \frac{1.3}{2.4}\cdot\frac{1}{5^2}\cdot\frac{2^5+1}{5^3} + \&c$

$= \frac{\pi}{2\sqrt{5}}\log_e\frac{4}{\sqrt{5}} + \frac{1}{\sqrt{5}}\left(\frac{1}{1^2}\cdot\frac{1}{2} - \frac{1}{3^2}\cdot\frac{1}{2^3} + \frac{1}{5^2}\cdot\frac{1}{2^5} - \&c\right)$

22. $\frac{\sin^2 x}{2^2} + \frac{1}{3}\cdot\frac{\sin^4 x}{4^2} + \frac{2.4}{3.5}\cdot\frac{\sin^6 x}{6^2} + \&c$

$$x+\cfrac{x^2+1}{2x+\cfrac{x^2+9}{2x+\cfrac{x^2+25}{2x+\&c}}}$$

$$=x+\cfrac{x^2-1}{2x+\cfrac{x^2-9}{2x+\cfrac{x^2-25}{2x+\&c}}}\times\frac{1-e^{-\pi x}}{1-2e^{-\frac{\pi x}{2}}\sin\frac{\pi x}{2}+e^{-\pi x}}$$

$$\frac{1+\frac{ax^2}{1-x}+\frac{a^2x^6}{(1-x)(1-x^2)}+\frac{a^3x^{12}}{(1-x)(1-x^2)(1-x^3)}+\&c}{1+\frac{ax}{1-x}+\frac{a^2x^4}{(1-x)(1-x^2)}+\frac{a^3x^9}{(1-x)(1-x^2)(1-x^3)}+\&c}$$

$$=\cfrac{1}{1+\cfrac{ax}{1+\cfrac{ax^2}{1+\cfrac{ax^3}{1+\cfrac{ax^4}{1+\&c}}}}}$$

$$\frac{1}{(1-ax)(1-ax^2)(1-ax^3)(1-ax^4)\,\&c}$$

$$=1+\frac{ax}{(1-x)(1-ax)}+\frac{a^2x^4}{(1-x)(1-x^2)(1-ax)(1-ax^2)}$$

$$+\frac{a^3x^9}{(1-x)(1-x^2)(1-x^3)(1-ax)(1-ax^2)(1-ax^3)}+\&c$$

$$\frac{f(-x^5,-x^{10})}{f(-x,-x^4)}=1+\frac{x}{1-x}+\frac{x^4}{(1-x)(1-x^2)}+\frac{x^9}{(1-x)(1-x^2)(1-x^3)}+\&c$$

$$\frac{f(-x^5,-x^{10})}{f(-x^2,-x^3)}=1+\frac{x^2}{1-x}+\frac{x^6}{(1-x)(1-x^2)}+\frac{x^{12}}{(1-x)(1-x^2)(1-x^3)}+\&c.$$

$$= \frac{1}{2}x^2 \log_e 2\sin x + \frac{x}{2}\left(\frac{\sin 2x}{1^2} + \frac{\sin 4x}{2^2} + \frac{\sin 6x}{3^2} + \&c\right)$$

$$+ \frac{1}{4}\left(\frac{\cos 2x}{1^3} + \frac{\cos 4x}{2^3} + \frac{\cos 6x}{3^3} + \&c\right)$$

Ex 1. $\frac{1}{2^2} + \frac{2}{3}\cdot\frac{1}{4^2} + \frac{2.4}{3.5}\cdot\frac{1}{6^2} + \frac{2.4.6}{3.5.7}\cdot\frac{1}{8^2} + \&c$

$$= \frac{\pi^2}{8}\log_e 2 - \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$$

2. $\frac{1}{2^2}\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{1}{4^2}\cdot\frac{1}{2^2} + \frac{2.4}{3.5}\cdot\frac{1}{6^2}\cdot\frac{1}{2^3} + \&c$

$$= \frac{\pi^2}{64}\log_e 2 + \frac{\pi}{8}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c\right) - \frac{5}{16}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$$

23. $\frac{\tan^2 x}{2^2} - \left(1+\frac{1}{3}\right)\frac{\tan^4 x}{4^2} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{\tan^6 x}{6^2} - \&c$

$$= \frac{x^2}{2}\log_e \tan x + x\left(\frac{\sin 2x}{1^2} + \frac{\sin 6x}{3^2} + \frac{\sin 10x}{5^2} + \&c\right)$$

$$+ \frac{1}{2}\left(\frac{\cos 2x}{1^3} + \frac{\cos 6x}{3^3} + \frac{\cos 10x}{5^3} + \&c\right) - \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$$

Ex. $\frac{1}{2^2} - \frac{1+\frac{1}{3}}{4^2} + \frac{1+\frac{1}{3}+\frac{1}{5}}{6^2} - \&c$

$$= \frac{\pi}{4}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c\right) - \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$$

24. $\frac{\cos^2 x + \sin^2 x}{2^2} + \frac{2}{3}\cdot\frac{\cos^4 x + \sin^4 x}{4^2} + \frac{2.4}{3.5}\cdot\frac{\cos^6 x + \sin^6 x}{6^2}$

$$+ \frac{2.4.6}{3.5.7}\cdot\frac{\cos^8 x + \sin^8 x}{8^2} + \&c$$

$$= -\frac{\pi^2}{8}\log_e 2\cos x + \frac{\pi}{2}\left\{\frac{\cos x}{1^2} + \frac{1}{2}\cdot\frac{\cos^3 x}{3^2} + \frac{1.3}{2.4}\cdot\frac{\cos^5 x}{5^2} + \&c\right\}$$

$$+ \frac{1}{4}\left\{\frac{\sin^2 2x}{2^2} + \frac{2}{3}\cdot\frac{\sin^4 2x}{4^2} + \frac{2.4}{3.5}\cdot\frac{\sin^6 2x}{6^2} + \&c\right\}$$

$$- \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$$

25. $\frac{\tan^2 x}{2^2} - \left(1+\frac{1}{3}\right)\frac{\tan^4 x}{4^2} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{\tan^6 x}{6^2} - \&c$

$$= 2\left\{\frac{\sin^2 x}{2^2} + \frac{2}{3}\cdot\frac{\sin^4 x}{4^2} + \frac{2.4}{3.5}\cdot\frac{\sin^6 x}{6^2} + \&c\right\}$$

$$- \frac{1}{4}\left\{\frac{\sin^2 2x}{2^2} + \frac{2}{3}\cdot\frac{\sin^4 2x}{4^2} + \frac{2.4}{3.5}\cdot\frac{\sin^6 2x}{6^2} + \&c\right\}$$

$$\int_0^\alpha \frac{d\phi}{\sqrt{(1-a\sin^2\phi)(1-b\sin^2\phi)}} = \frac{1}{\sqrt{1-b}}\int_0^\beta \frac{d\phi}{\sqrt{1-\frac{a-b}{1-b}\sin^2\phi}}$$

c $\quad \dfrac{\tan\beta}{\tan\alpha} = \sqrt{1-b}$.

(i) $\sqrt[24]{y}\,(1-y)(1-y^2)(1-y^3)\,\&c = \dfrac{\sqrt[24]{x}\,\sqrt[8]{1-x}}{\sqrt[8]{3}}\,\sqrt{1+\frac{1.2}{3^2}x+\&c}$

(i) $= \dfrac{\sqrt[24]{x}\,\sqrt[12]{1-x}}{\sqrt[4]{2}}\sqrt{1+\frac{1.3}{4^2}x+\&c}$. (iii) $= \sqrt[24]{\dfrac{x(1-x)}{432}}\sqrt{1+\frac{1.5}{6^2}x+\&c}$

$\sqrt[12]{y}\,(1-y^2)(1-y^4)(1-y^6)\,\&c = \dfrac{\sqrt[12]{x}\,\sqrt[24]{1-x}}{\sqrt{2}}\sqrt{1+\frac{1.3}{4^2}x+\&c}$

$\sqrt[8]{y}\,(1-y^3)(1-y^6)(1-y^9)\,\&c = \sqrt[8]{\dfrac{x}{27}}\,\sqrt[24]{1-x}\,\sqrt{1+\frac{1.2}{3^2}x+\&c}$

$$1+240\left(\frac{1^3y^3}{1-y^3}+\frac{2^3y^6}{1-y^6}+\&c\right)=\left(1+\frac{1.2}{3^2}x+\&c\right)^4\left(1-\frac{8}{9}x\right).$$

$$1-504\left(\frac{1^5y^3}{1-y^3}+\frac{2^5y^6}{1-y^6}+\&c\right)=\left(1+\frac{1.2}{3^2}x+\&c\right)^6\left(1-\frac{4}{3}x+\frac{8}{27}x^2\right).$$

26. i. $\int_0^{\frac{\pi}{2}} x \cos^n x \sin nx\, dx = \frac{\pi}{2^{n+2}}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$

ii. $\int_0^{\frac{\pi}{2}} \cos^n x \sin nx\, dx = \frac{1}{2^{n+1}}\left(\frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^n}{n}\right)$

These are true for all values n.

Cor. 1. Show that $\frac{2-1}{1} + \frac{2^2-1}{2} + \frac{2^3-1}{3} + \cdots + \frac{2^x-1}{x}$ can be expanded in ascending powers of x in a convergent series the first two terms being $\frac{S_2}{2}x + \frac{S_3}{8}x^2 + \&c$

2. If $\phi(x) = \frac{2-1}{1} + \frac{2^2-1}{2} + \cdots + \frac{2^x-1}{x}$, then show that

$\phi(-x) + \Sigma\frac{1}{x-1} + \frac{1}{2^x}\frac{1}{x} + \frac{1}{2^{2x}}\frac{1}{1+x} + \frac{1}{2^{3x}}\frac{1}{2+x} + \&c$

hence show how to find the value of the series

$\frac{1}{1^n}\frac{1}{2} + \frac{1}{2^n}\frac{1}{2^2} + \frac{1}{3^n}\frac{1}{2^3} + \&c$

27. $1^r\log_e 1 + 2^r\log_e 2 + 3^r\log_e 3 + \cdots + x^r\log_e x = \phi_r(x)$

$\phi_r(x) = C_r + (1^r + 2^r + 3^r + \cdots + x^r - S_{-r})\log_e x - \frac{x^{r+1}}{(r+1)^2}$

$+ \frac{B_2}{\lfloor 2}\, r\,\frac{1}{r}\, x^{r-1} - \frac{B_4}{\lfloor 4}\, r(r-1)(r-2)\left(\frac{1}{r} + \frac{1}{r-1} + \frac{1}{r-2}\right)x^{r-3} + \&c$

and $C_{r-1} = \frac{B_r}{r}\left\{\cos\frac{\pi r}{2}\left(\Sigma\frac{1}{r-1} - C_0 - \log_e 2\pi\right) + \frac{\pi}{2}\sin\frac{\pi r}{2}\right\}$

$- \frac{2\lfloor r-1}{(2\pi)^r}\cos\frac{\pi r}{2}\left\{\frac{\log_e 1}{1^r} + \frac{\log_e 2}{2^r} + \frac{\log_e 3}{3^r} + \&c\right\}$

Cor. If r is even $C_r = -\frac{\pi}{2}\frac{B_{r+1}}{r+1}\cos\frac{\pi r}{2} = \pm\frac{\lfloor r}{2(2\pi)^r}S_{r+1}$

$C_0 = \frac{1}{2}\log_e 2\pi$, $C_2 = -\frac{S_3}{4\pi^2}$, $C_4 = \frac{3S_5}{4\pi^4}$, $C_6 = -\frac{45S_7}{8\pi^6}$ &c

Sol. Divide both sides in IX 1 by x^n and then differentiate both sides with regards to n

Ex. 1. $\dfrac{(1^1.2^2.3^3\ldots x^x)^2 e^{\frac{1}{6}(x^2-\Sigma\frac{1}{x})}}{\sqrt[3]{\dfrac{x^{x(x+3)}\lfloor x^2}{x+\frac{1}{4}}}}$ when $x=\infty$

$= 1^{\frac{1}{\pi^2}}.2^{\frac{1}{(2\pi)^2}}.3^{\frac{1}{(3\pi)^2}}.4^{\frac{1}{(4\pi)^2}}.5^{\frac{1}{(5\pi)^2}}$ &c

$$\phi(1) - \phi(2) + \phi(3) - \&c$$
$$= \phi(0) - \phi(-1) + \phi(-2) - \&c$$

If $\alpha\beta = \frac{\pi}{4}$, then

$$\sqrt{\alpha}\left\{1 - \frac{\alpha^2}{\lfloor 2} + \frac{5}{\lfloor 4}\alpha^4 - \frac{61}{\lfloor 6}\alpha^6 + \&c\right\}$$
$$= \sqrt{\beta}\left\{1 - \frac{\beta^2}{\lfloor 2} + \frac{5}{\lfloor 4}\beta^4 - \frac{61}{\lfloor 6}\beta^6 + \&c\right\}$$

~~$\sqrt{\alpha}\left\{e^{\alpha^2} - e^{9\alpha^2} + e^{25\alpha^2} - \&c\right\} = \sqrt{\beta}\left\{e^{\beta^2} - e^{9\beta^2} + e^{25\beta^2} - \&c\right\}$~~

~~$\int_0^\infty e^{x^2} \sin 2nx\, dx = \frac{\sqrt{\pi}}{2} e^{n^2}$.~~

If $\alpha\beta = \pi$, then

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}}{e^{\alpha x} + e^{-\alpha x}}\,dx = \sqrt{\beta}\int_0^\infty \frac{e^{-x^2}}{e^{\beta x} + e^{-\beta x}}\,dx$$

$$1 - xa + x^3a^2 - x^6a^3 + x^{10}a^4 - \&c$$
$$= \cfrac{1}{1 + \cfrac{xa}{1 + \cfrac{(x^2 - x)a}{1 + \cfrac{x^3a}{1 + \cfrac{(x^4 - x^2)a}{1 + \cfrac{x^5a}{1 + \&c}}}}}}$$

$$D_{2n} = 1 + ax^n \cdot \frac{1 - x^n}{1 - x} + a^2x^{2n} \cdot \frac{(1 - x^n)(1 - x^{n-1})}{(1 - x)(1 - x^2)}$$
$$+ a^3x^{3n} \cdot \frac{(1 - x^n)(1 - x^{n-1})(1 - x^{n-2})}{(1 - x)(1 - x^2)(1 - x^3)} + \&c = \phi(a)$$

$$D_{2n+1} = \phi(ax).$$

2. $\left\{\left(\frac{1}{x}\right)^{1^2}\left(\frac{2}{x}\right)^{2^2}\left(\frac{3}{x}\right)^{3^2}\left(\frac{4}{x}\right)^{4^2}\cdots\left(\frac{x}{x}\right)^{x^2}\right\} e^{\frac{x^3}{9}-\frac{x}{12}}$ when $x=\infty$

$= e^{\frac{2}{\pi^2}\left(\frac{1}{2^3}+\frac{1}{4^3}+\frac{1}{6^3}+\frac{1}{8^3}+\&c\right)}$

28. $\phi_n(x) + nC_{n-1}x + \frac{n(n-1)}{\lfloor 2}C_{n-2}x^2 + \frac{n(n-1)(n-2)}{\lfloor 3}C_{n-3}x^3 + \cdots$

$+ C_0x^n + S_1\frac{x^{n+1}}{n+1}\lfloor n - S_2\frac{x^{n+2}\lfloor n}{(n+1)(n+2)} + \&c = f(x,n)$

where C_n is the constant of $1^n\log_e 1 + 2^n\log_e 2 + \&c$

& S_n is that of $\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\&c$

$$f(x,n) = \frac{1^n+2^n+3^n+\cdots+x^n}{n} + n\int_0^x f(x,n-1)\,dx$$

$$f(x,n) = (1^n+2^n+\cdots x^n - S_{-n})\,\Sigma\frac{1}{n} - \frac{n}{\lfloor 2}B_2x^{n-1} + \frac{n(n-1)(n-2)}{\lfloor 4}B_4x^{n-3}\left(1+\frac{1}{2}+\frac{1}{3}\right) - \frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 6}B_6\,x^{n-5}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right) + \&c$$

29. $\phi_n(x) - n^n\left\{\phi_n\left(\frac{x}{n}\right)+\phi_n\left(\frac{x-1}{n}\right)+\cdots+\phi_n\left(\frac{x-n+1}{n}\right)\right\}$

$=(1^n+2^n+\cdots x^n)\log_e n - S_{-n}\log_e n - (n^{n+1}-1)C_n$

Cor 1. $\phi_n\left(-\frac{1}{n}\right)+\phi_n\left(-\frac{2}{n}\right)+\phi_n\left(-\frac{3}{n}\right)+\cdots+\phi_n\left(-\frac{n-1}{n}\right)$

$=\frac{\log_e n}{n^n}S_{-n} + \left(n-\frac{1}{n^n}\right)C_n$

Cor 2. $\phi_n\left(-\frac{1}{2}\right) = \frac{\log_e 2}{2^n}S_{-n} + \left(2-\frac{1}{2^n}\right)C_n$.

30. If n is even

$\phi_n(x-1)+\phi_n(-x) = 2C_n + \frac{\lfloor n}{(2\pi)^n}\cos\frac{\pi n}{2}\left\{\frac{\cos 2\pi x}{1^{n+1}}+\frac{\cos 4\pi x}{2^{n+1}}\right.$

If n is odd

$\phi_n(x-1)-\phi_n(-x) = \frac{\lfloor n}{(2\pi)^n}\sin\frac{\pi n}{2}\left\{\frac{\sin 2\pi x}{1^{n+1}}+\frac{\sin 4\pi x}{2^{n+1}}\right.$

Sol. $\Sigma\frac{1}{x-1} - \Sigma\frac{1}{-x} = -\pi\cot\pi x = -\frac{\pi}{2}(\sin 2\pi x + \sin 4\pi x$

Integrate both sides $n+1$ times.

If $\int_0^h \phi(x)\cos nx\,dx = \psi(n)$ & $\alpha\beta = \frac{\pi}{2}$

then $\alpha\left\{\frac{1}{2}\phi(0) + \phi(\alpha)\cos n\alpha + \phi(2\alpha)\cos 2n\alpha + \ldots\ldots + \phi(m\alpha)\cos mn\alpha.\right\}$

$= \psi(n) + \psi(2\beta - n) + \psi(2\beta + n) + \psi(4\beta - n) + \psi(4\beta + n) + \psi(6\beta - n) + \psi(6\beta + n) +$ &c ad. inf.

where $m\alpha$ is the greatest multiple of α less than h; but if h be a multiple of α the last term is $\frac{1}{2}\phi(h)\cos nh$; in both the cases n lies between 0 & 2β.

$\int_0^h \frac{\sin nx}{\sin x}\phi(x)\,dx$

$= \pi\left\{\frac{1}{2}\phi(0) - \phi(\pi)\cos n\pi + \phi(2\pi)\cos 2n\pi - \ldots\ldots \pm \phi(m\pi)\cos mn\pi.\right\}$

$- 2\psi(n+1) - 2\psi(n+3) - 2\psi(n+5) -$ &c ad. inf.

N. B. More general theorems true for all values of r can be got by differentiating IX 14 & 15 with respect to r.

31. If $1 \log_e 1 + 2\log_e 2 + \cdots + x\log_e x = \phi_1(x)$, then

i. $\pi\{\phi_1(x-1) - \phi_1(-x)\} = \psi(x) - \pi x \log_e(2\sin \pi x)$ where

$$\psi(x) = \sin \pi x + \frac{1}{2}\cdot\frac{\sin^3 \pi x}{3^2} + \frac{1.3}{2.4}\cdot\frac{\sin^5\pi x}{5^2} + \&c$$

$$= \tan\pi x - \left(1+\frac{1}{3}\right)\frac{\tan^3\pi x}{3} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{\tan^5\pi x}{5} - \&c$$

ii. $\psi(x) + \psi(\frac{1}{2} - x) = \tan \pi x - \frac{\tan^3 \pi x}{3^2} + \frac{\tan^5\pi x}{5^2} - \&c$
$+ \frac{\pi}{2}\log_e(2\cos\pi x)$

iii. $\psi(\frac{1}{2} - x) + \frac{1}{2}\psi(2x) - \psi(x) = \frac{\pi}{2}\log_e(2\cos\pi x)$.

Ex. 1. $\psi(\frac{1}{2}) = \frac{\pi}{2}\log_e 2$.

2. $\psi(\frac{1}{4}) = \left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c\right) + \frac{\pi}{4}\log_e 2$.

3. $\psi(\frac{1}{6}) = \frac{3\sqrt{3}}{4}\left(\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} + \&c\right) - \frac{\pi^2}{6\sqrt{3}}$

4. $\psi(\frac{1}{3}) = \frac{\sqrt{3}}{2}\left(\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} + \&c\right) - \frac{\pi^2}{9\sqrt{3}} + \frac{\pi}{6}\log_e 3$.

5. $\psi(\frac{1}{2} - x) + \psi(\frac{1}{2} + x) = \pi\log_e(2\cos \pi x)$

6. Find $\psi(\frac{2}{3})$, $\psi(\frac{1}{4})$, $\psi(\frac{1}{6})$, $\psi(\frac{3}{8}) - \psi(\frac{1}{8})$

7. Find $\psi(\frac{1}{8})$ & $\psi(\frac{1}{12})$

8. $2\psi(x) - \frac{1}{2}\psi(2x) = \tan \pi x - \frac{\tan^3\pi x}{3^2} + \frac{\tan^5 \pi x}{5^2} - \&c$

Similarly we can find peculiarities for $\phi_2(x)$, $\phi_3(x)$ &c

32. $\sin 2x + \frac{2}{3^2}\sin^3 2x + \frac{2.4}{3.5^2}\sin^5 2x + \frac{2.4.6}{3.5.7^2}\sin^7 2x + \&c$

$$= 2\left(\frac{\tan x}{1^2} - \frac{\tan^3 x}{3^2} + \frac{\tan^5 x}{5^2} - \&c\right)$$

Cor. $\frac{2x}{1+x^2} + \frac{2}{3^2}\left(\frac{2x}{1+x^2}\right)^3 + \frac{2.4}{3.5^2}\left(\frac{2x}{1+x^2}\right)^5 + \&c$

$$= 2\left(\frac{x}{1^2} - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \&c\right)$$

If $\dfrac{a_n}{b_n x + \dfrac{a_{n+1}}{b_{n+1} x + \dfrac{a_{n+2}}{b_{n+2} x + \&c}}} = C_n\left(1 - P_n x + Q_n x^2 - R_n x^3 + \&c\right)$

then $C_n C_{n+1} = a_n$; $P_n + P_{n+1} = \dfrac{b_n}{C_{n+1}}$ or $\dfrac{b_n C_n}{a_n}$.

$Q_n + Q_{n+1} = (P_n)^2$. $R_n + R_{n+1} = P_n(Q_n - Q_{n+1})$.

$S_n + S_{n+1} = P_n(R_n - R_{n+1}) - Q_n Q_{n+1}$.

Generally

$Z_n + Z_{n+1} = P_n(Y_n - Y_{n+1}) - Q_n X_{n+1} - R_n W_{n+1}$

$- S_n V_{n+1} - \ldots\ldots - X_n Q_{n+1}$.

If $\displaystyle\int_0^\alpha \frac{d\theta}{\sqrt{1 - x\sin^2\theta}} + \int_0^\beta \frac{d\theta}{\sqrt{1 - x\sin^2\theta}} = \int_0^\gamma \frac{d\theta}{\sqrt{1 - x\sin^2\theta}}$

then $\tan\dfrac{\gamma}{2} = \dfrac{\sin\alpha\sqrt{1 - x\sin^2\beta} + \sin\beta\sqrt{1 - x\sin^2\alpha}}{\cos\alpha + \cos\beta}$,

or $\cot\alpha\cot\beta = \dfrac{\cos\gamma}{\sin\alpha\sin\beta} + \sqrt{1 - x\sin^2\gamma}$

$\dfrac{\sqrt{x}}{2} = \dfrac{\sqrt{\sin\delta\sin(\delta-\alpha)\sin(\delta-\beta)\sin(\delta-\gamma)}}{\sin\alpha\sin\beta\sin\gamma}$ where $2\delta = \alpha + \beta + \gamma$.

Ex. 1. $1+\frac{2}{3^2}+\frac{2.4}{3.5^2}+\frac{2.4.6}{3.5.7^2}+\&c = 2\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\&c\right)$

2. $1+\frac{2}{3^2}\cdot\frac{3}{4}+\frac{2.4}{3.5^2}\left(\frac{3}{4}\right)^2+\frac{2.4.6}{3.5.7^2}\left(\frac{3}{4}\right)^3+\&c$

$= -\frac{\pi}{3\sqrt{3}}\log_e 3 - \frac{10}{27}\pi^2 + 5\left(\frac{1}{1^2}+\frac{1}{4^2}+\frac{1}{7^2}+\&c\right)$

3. $\frac{1}{2}+\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2.4}{3.5^2}\cdot\frac{1}{2^5}+\frac{2.4.6}{3.5.7^2}\cdot\frac{1}{2^7}+\&c$

$= -\frac{\pi}{6}\log_e(2+\sqrt{3})+\frac{4}{3}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\&c\right)$

4. $1+\frac{2}{3^2}\cdot\frac{1}{2}+\frac{2.4}{3.5^2}\cdot\frac{1}{2^2}+\frac{2.4.6}{3.5.7^2}\cdot\frac{1}{2^3}+\&c$

$= -\frac{\pi}{2\sqrt{2}}\log_e(1+\sqrt{2})-\frac{\pi^2}{4\sqrt{2}}+4\left(\frac{1}{1^2}-\frac{1}{5^2}+\frac{1}{9^2}-\frac{1}{13^2}+\&c\right)$

5. $\left(1-\frac{3}{4}\right)+\frac{2}{3^2}\left(1-\frac{3}{4^2}\right)+\frac{2.4}{3.5^2}\left(1-\frac{3}{4^3}\right)+\frac{2.4.6}{3.5.7^2}\left(1-\frac{3}{4^4}\right)+\&c$

$= \frac{\pi}{4}\log_e(2+\sqrt{3})$

6. $\frac{2x}{1-x^2}-\frac{2}{3^2}\left(\frac{2x}{1-x^2}\right)^3+\frac{2.4}{3.5^2}\left(\frac{2x}{1-x^2}\right)^5-\&c$

$= 2\left(\frac{x}{1^2}+\frac{x^3}{3^2}+\frac{x^5}{5^2}+\frac{x^7}{7^2}+\&c\right)$

7. $1-\frac{2}{3^2}+\frac{2.4}{3.5^2}-\frac{2.4.6}{3.5.7^2}+\&c = \frac{\pi^2}{8}-\frac{1}{2}\left\{\log_e(1+\sqrt{2})\right\}^2$

8. $\frac{1}{2}-\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2.4}{3.5^2}\cdot\frac{1}{2^5}-\frac{2.4.6}{3.5.7^2}\cdot\frac{1}{2^7}+\&c$

$= \frac{\pi^2}{12}-\frac{3}{2}\left(\log\frac{\sqrt{5}+1}{2}\right)^2$

9. $\tan 2x-\frac{2}{3^2}\tan^3 2x+\frac{2.4}{3.5^2}\tan^5 2x-\&c$

$= 2\left(\tan x+\frac{\tan^3 x}{3^2}+\frac{\tan^5 x}{5^2}+\&c\right)$

10. $\left(\frac{x}{1+x}\right)+\frac{1}{3^2}\left(\frac{x}{1+x}\right)^2+\frac{1}{5^2}\left(\frac{x}{1+x}\right)^3+\frac{1}{7^2}\left(\frac{x}{1+x}\right)^4+\&c$

$= x-\frac{2}{3}\left(1+\frac{1}{3}\right)x^2+\frac{2.4}{3.5}\left(1+\frac{1}{3}+\frac{1}{5}\right)x^3-\&c$

If $\frac{\tan\alpha}{\tan\beta} = \sqrt{1-x}$, then

$$\int_0^\alpha \frac{d\theta}{\sqrt{1-x\cos^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-x\sin^2\theta}}.$$

$$\frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1+x\sin\theta}} = \int_0^{\frac{\pi}{2}} \frac{\cos^{-1}(x\sin^2\theta)}{\sqrt{1-x^2\sin^4\theta}}\,d\theta.$$

$$\left\{\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-x\sin^2\theta}}\right\}^2 = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta\, d\phi}{\sqrt{(1-x\sin^2\theta)(1-x\sin^2\phi\sin^2\theta)}}$$

If $z = 1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c$

& $y = \pi\,\frac{1+\left(\frac{1}{2}\right)^2(1-x) + \&c}{1+\left(\frac{1}{2}\right)^2 x + \&c}$

then $\frac{1}{2} + \frac{\operatorname{sech} y}{1+n^2} + \frac{\operatorname{sech} 2y}{1+(2n)^2} + \frac{\operatorname{sech} 3y}{1+(3n)^2} + \&c$

$$= \cfrac{z}{2 + \cfrac{(nz)^2 x}{2 + \cfrac{(2nz)^2}{2 + \cfrac{(3nz)^2 x}{2 + \cfrac{(4nz)^2}{2 + \&c}}}}}$$

$$\int_0^\infty e^{-n\int_0^\phi \frac{d\theta}{\sqrt{1-x\sin^2\theta}}}\,d\phi = \cfrac{1}{n + \cfrac{x}{n + \cfrac{4}{n + \cfrac{9x}{n + \&c}}}}$$

If $\frac{\sin\alpha}{\sin\beta} = \sqrt{x}$, then

$$\int_0^\alpha \frac{d\theta}{\sqrt{x-\sin^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-x\sin^2\theta}}$$

1. If x, y, z, u & n are positive integers, then

$$n\frac{\lfloor x+n \lfloor y+n \lfloor z+n \lfloor u+n \, \lfloor x+y+z+n \, \lfloor y+z+u+n \, \lfloor z+u+x+n \, \lfloor u+x+y+n}{\lfloor n \, \lfloor x+y+n \, \lfloor y+z+n \, \lfloor z+u+n \, \lfloor u+x+n \, \lfloor x+z+n \, \lfloor y+u+n \, \lfloor x+y+z+u+n}$$

$$= n - (n+2)\frac{n}{\lfloor 1}\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}\cdot\frac{u}{u+n+1}\cdot\frac{x+y+z+u+2n+1}{x+y+z+u+n}$$

$$+ (n+4)\cdot\frac{n(n+1)}{\lfloor 2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\cdot\frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$$\times\frac{u(u-1)}{(u+n+1)(u+n+2)}\cdot\frac{(x+y+z+u+2n+1)(x+y+z+u+2n+2)}{(x+y+z+u+n)(x+y+z+u+n+1)}$$

$- \&c$

N. B. The above result is not true for all values of x, y, z, u
and n. For example it is not true when $x+y+z+u+n =$
x, y, z or u owing to the extraneous factor containing
all the quantities x, y, z & u in each term. Unless we
get rid of this factor identities deduced from the
above won't be true for all values. The only way
to get rid of this is to make u infinitely great.
The solution of this theorem is evident from the result.

2. $$\Sigma\frac{1}{x+n} + \Sigma\frac{1}{y+n} + \Sigma\frac{1}{z+n} - \Sigma\frac{1}{x+y+n} - \Sigma\frac{1}{y+z+n} -$$

$$\Sigma\frac{1}{z+x+n} + \Sigma\frac{1}{x+y+z+n} - \Sigma\frac{1}{n}$$

$$= \left(1+\frac{1}{n+1}\right)\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}\cdot\frac{x+y+z+2n+1}{x+y+z+n}$$

$$+ \left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\cdot\frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$$\times\frac{(x+y+z+2n+1)(x+y+z+2n+2)}{(x+y+z+n)(x+y+z+n+1)} + \&c$$

This is true only for positive integral values.

Sol. Subtract both sides in XII 1 from n then divide both sides by u and then put $u = 0$.

3. If x, y, z, and n are positive integers, then [illegible]

If $\dfrac{\sin\beta}{\sin\alpha} = \dfrac{1+x}{1+x\sin^2\alpha}$,

then
$$(1+x)\int_0^\alpha \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\theta}}$$

$$(1+2x)\int_0^\alpha \frac{d\theta}{\sqrt{1-x^3\frac{2+x}{1+2x}\sin^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-x\left(\frac{2+x}{1+2x}\right)^3\sin^2\theta}}$$

if $\dfrac{1+\sin\beta}{1-\sin\beta} = \dfrac{1+\sin\alpha}{1-\sin\alpha}\cdot\left(\dfrac{1+x\sin\alpha}{1-x\sin\alpha}\right)^2$.

$$(1+x)^2\int_0^\alpha \frac{d\theta}{\sqrt{1-x^4\sin^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-\left\{1-\left(\frac{1-x}{1+x}\right)^4\right\}\sin^2\theta}}$$

if $\dfrac{1+\sin\beta}{1-\sin\beta} = \dfrac{1+\sin\alpha}{1-\sin\alpha}\left(\dfrac{1+x\sin\alpha}{1-x\sin\alpha}\right)^2\dfrac{1+x^2\sin\alpha}{1-x^2\sin\alpha}$,

---

If $\sin(2\beta-\alpha) = x\sin\alpha$

$$(1+x)\int_0^\alpha \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = 2\int_0^\beta \frac{d\theta}{\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\theta}}$$

If $\dfrac{\tan\alpha}{\tan\beta} = \sqrt{1+x}$

$$\int_0^\alpha \frac{d\theta}{\sqrt{1+x\cos^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-x^2\sin^4\theta}}.$$

$$\frac{\lfloor\underline{n}\,\lfloor\underline{x+y+n}\,\lfloor\underline{y+z+n}\,\lfloor\underline{z+x+n}}{\lfloor\underline{x+n}\,\lfloor\underline{y+n}\,\lfloor\underline{z+n}\,\lfloor\underline{x+y+z+n}} = 1 + \frac{xyz}{\lfloor\underline{1}\,(n+1)(x+y+z+n)}$$

$$+ \frac{x(x-1)\,y(y-1)\,z(z-1)}{\lfloor\underline{2}\,(n+1)(n+2)(x+y+z+n)(x+y+z+n-1)} + \&c$$

Sol. Divide both sides in XII/ by n, write $-n+m$ for u and then make n infinitely great.

4 If $x, y, z,$ and n are positive integers, then

$$\frac{(x+n)(y+n)(z+n)(x+y+z+n)}{(x+y+n)(y+z+n)(z+x+n)} = n +$$

$$(n+2)\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}\cdot\frac{x+y+z+2n}{x+y+z+n-1} +$$

$$(n+4)\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\cdot\frac{z(z-1)}{(z+n+1)(z+n+2)} \times$$

$$\frac{(x+y+z+2n)(x+y+z+2n+1)}{(x+y+z+n-1)(x+y+z+n-2)} + \&c$$

Sol. Put $u = -1$ in XII/.

Ex. If x is a positive integer shew that

1. $$\frac{(\lfloor\underline{x}\,\lfloor\underline{3x-2})^4}{(\lfloor\underline{2x-1})^6\,\lfloor\underline{4x-3}} = 1 - 3\left(\frac{x-1}{x+1}\right)^4\frac{4x-1}{4x-3} + 5\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^4\frac{4x-1}{4x-3}\cdot\frac{4x}{4x-4}$$
$$-7\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\cdot\frac{x-3}{x+3}\right)^4\frac{4x-1}{4x-3}\cdot\frac{4x}{4x-4}\cdot\frac{4x+1}{4x-5} + \&c.$$

2. $$\frac{3}{2}\leq\frac{1}{\frac{x-1}{2}} - \frac{3}{2}\leq\frac{1}{x-1} + \frac{1}{2}\leq\frac{1}{\frac{3x-3}{2}}$$
$$= 1\left(\frac{x-1}{x+1}\right)^3\frac{3x-1}{3x-3} + \frac{1}{2}\left(\frac{x-1}{x+1}\cdot\frac{x-3}{x+3}\right)^3\frac{3x-1}{3x-3}\cdot\frac{3x+1}{3x-5} + \&c$$

3. $$\left(\frac{x}{2x-1}\right)^3(3x-2) = 1 + 3\left(\frac{x-1}{x+1}\right)^3\frac{3x-1}{3x-3} + 5\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^3\frac{3x-1}{3x-3}\cdot\frac{3x}{3x-4}$$
$$+ \&c.$$

4. $$\left(\frac{\lfloor\underline{2x}}{\lfloor\underline{x}}\right)^3\frac{1}{\lfloor\underline{3x}} = 1 + \left(\frac{x}{\lfloor\underline{1}}\right)^2\frac{3x}{3x} + \left\{\frac{x(x-1)}{\lfloor\underline{2}}\right\}^2\frac{x(x-1)}{3x(3x-1)} + \&c$$

5. $$\frac{8}{9}\cdot\left(\frac{\lfloor\underline{3x}}{\lfloor\underline{2x}}\right)^3\frac{\lfloor\underline{x}}{\lfloor\underline{4x}} = 1 + \frac{x}{\lfloor\underline{1}}\cdot\frac{x-1}{x+1}\cdot\frac{x}{4x-1} + \frac{x(x-1)}{\lfloor\underline{2}}\cdot\frac{(x-1)(x-2)}{(x+1)(x+2)}$$
$$\times\frac{x(x-1)}{(4x-1)(4x-2)} + \&c$$

$$x + (x+2)\frac{\alpha}{x-\alpha+1}\cdot\frac{\beta}{x-\beta+1} + (x+4)\frac{\alpha(\alpha+1)}{(x-\alpha+1)(x-\alpha+2)}$$

$$\times\frac{\beta(\beta+1)}{(x-\beta+1)(x-\beta+2)} + \text{\&c to } k+1 \text{ terms}$$

$$= \frac{1}{x-\alpha-\beta}\left\{(x-\alpha)(x-\beta) - \frac{\lfloor\!\underline{x-\alpha}\;\lfloor\!\underline{x-\beta}}{\lfloor\!\underline{\alpha-1}\;\lfloor\!\underline{\beta-1}}\cdot\frac{\lfloor\!\underline{\alpha+k}\;\lfloor\!\underline{\beta+k}}{\lfloor\!\underline{x-\alpha+k}\;\lfloor\!\underline{x-\beta+k}}\right\}$$

Ex. If $\alpha+\beta+\gamma+1=k$, then

$$(k+1)\frac{\lfloor\!\underline{k}}{\lfloor\!\underline{0}}\cdot\frac{\lfloor\!\underline{\alpha}\;\lfloor\!\underline{\beta}\;\lfloor\!\underline{\gamma}}{\lfloor\!\underline{k-\alpha}\;\lfloor\!\underline{k-\beta}\;\lfloor\!\underline{k-\gamma}} + (k+3)\frac{\lfloor\!\underline{k+1}}{\lfloor\!\underline{1}}\cdot\frac{\lfloor\!\underline{\alpha+1}\;\lfloor\!\underline{\beta+1}\;\lfloor\!\underline{\gamma+1}}{\lfloor\!\underline{k-\alpha+1}\;\lfloor\!\underline{k-\beta+1}\;\lfloor\!\underline{k-\gamma+1}}$$

$$+ (k+5)\frac{\lfloor\!\underline{k+2}}{\lfloor\!\underline{2}}\cdot\frac{\lfloor\!\underline{\alpha+2}\;\lfloor\!\underline{\beta+2}\;\lfloor\!\underline{\gamma+2}}{\lfloor\!\underline{k-\alpha+2}\;\lfloor\!\underline{k-\beta+2}\;\lfloor\!\underline{k-\gamma+2}} + \text{\&c to } n \text{ terms}$$

$- 2\log_e n$ when n becomes infinite.

$$= -\Sigma\frac{1}{\alpha} - \Sigma\frac{1}{\beta} - \Sigma\frac{1}{\gamma} + 2C.$$

5. For all values of x, y, z and n,

$$n\cdot\frac{\lfloor x+n \,\lfloor y+n \,\lfloor z+n \,\lfloor x+y+z+n}{\lfloor n \,\lfloor x+y+n \,\lfloor y+z+n \,\lfloor z+x+n} = n-(n+2)\frac{n}{\lfloor 1}\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}$$

$$+(n+4)\frac{n(n+1)}{\lfloor 2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\cdot\frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$- \&c.$

Sol. Make a infinite in XII 1.

6. $$\Sigma\frac{1}{x+n}+\Sigma\frac{1}{y+n}-\Sigma\frac{1}{x+y+n}-\Sigma\frac{1}{n} = \left(1+\frac{1}{n+1}\right)\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}$$

$$+\left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}+\&c$$

Sol. Subtract both sides in XII 5 from n, divide both sides by z and then put $z=0$.

7. $$\frac{(x+n)(y+n)}{x+y+n} = n+(n+2)\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}+$$

$$(n+4)\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}+\&c$$

Sol. Put $z=-1$ in XII 5.

8. $$n\,\frac{\lfloor x+y}{\lfloor x\,\lfloor y}\cdot\frac{\lfloor x+n\,\lfloor y+n}{\lfloor n\,\lfloor x+y+n} = n+(n+2)\frac{n^2}{(\lfloor 1)^2}\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}$$

$$+(n+4)\frac{n^2(n+1)^2}{(\lfloor 2)^2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}+\&c$$

Sol. Write $-n$ for z in XII 5.

9. $$\frac{\lfloor x+n\,\lfloor y+n}{\lfloor n\,\lfloor x+y+n}\cdot\frac{\lfloor \frac{n}{2}\,\lfloor x+y+\frac{n}{2}}{\lfloor x+\frac{n}{2}\,\lfloor y+\frac{n}{2}} = 1+\frac{n}{\lfloor 1}\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}$$

$$+\frac{n(n+1)}{\lfloor 2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}+\&c$$

Sol. Put $z=-\frac{n}{2}$ in XII 5.

N.B. This result is very important as the isolated factors n, $n+2$, $n+4$ disappear.

10. $$n\,\frac{\lfloor x+n\,\lfloor y+n}{\lfloor n\,\lfloor x+y+n}\cdot\frac{\lfloor \frac{n-1}{2}\,\lfloor x+y+\frac{n-1}{2}}{\lfloor x+\frac{n-1}{2}\,\lfloor y+\frac{n-1}{2}} = n+(n+2)\frac{n}{\lfloor 1}\cdot\frac{xy}{(x+n+1)(y+n+1)}$$

$$(1+x)\int_0^{\alpha} \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = 2\int_0^{\beta} \frac{d\theta}{\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\theta}}$$

if $(1+x)\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\beta} = \sqrt{1-x^2\sin^2\alpha} + x\cos\alpha$.

$$\int_0^{\alpha} \frac{d\theta}{\sqrt{1-x\sin^2\theta}} + \int_0^{\beta} \frac{d\theta}{\sqrt{1-x\sin^2\theta}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-x\sin^2\theta}}$$

if $(1-x\sin^2\alpha)(1-x\sin^2\beta) = 1-x$.

if $\cot\alpha.\cot\beta = \sqrt{1-x}$.

$$\int_0^{\alpha} \frac{d\theta}{\sqrt{1-x\sin^2\theta}} = 2\int_0^{\beta} \frac{d\theta}{\sqrt{1-x\sin^2\theta}}$$

if $\frac{\tan\frac{\alpha}{2}}{\tan\beta} = \sqrt{1-x\sin^2\beta}$.

$$\int_0^{\sin^{-1}\sqrt[8]{\frac{\beta}{\alpha^3}}} \frac{d\theta}{\sqrt{1-\alpha\sin^2\theta}} = \frac{\pi}{6}\left\{1+\left(\tfrac{1}{6}\right)^2\alpha + \quad\right\}$$

where $\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)} = 1$.

$+(n+4)\frac{n(n+1)}{\underline{|2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}+\&c$

11. $n\frac{\underline{|x+n}\,\underline{|y+n}}{\underline{|n}\,\underline{|x+y+n}} = n-(n+2)\frac{n}{\underline{|1}}\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}+(n+4)\frac{n(n+1)}{\underline{|2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\ \&c$

12. $\left\{\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\frac{1}{(n+3)^2}+\&c\right\}-\left\{\frac{1}{(x+n+1)^2}+\frac{1}{(x+n+2)^2}+\&c\right\}$
$=\left(1+\frac{1}{n+1}\right)\frac{1}{n+1}\cdot\frac{x}{x+n+1}-\left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{\underline{|2}}{(n+1)(n+2)}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}$
$+\left(\frac{1}{3}+\frac{1}{n+3}\right)\frac{\underline{|3}}{(n+1)(n+2)(n+3)}\cdot\frac{x(x-1)(x-2)}{(x+n+1)(x+n+2)(x+n+3)}-\&c$

Sol. Divide both sides in XII 6 by y and then put y = 0.

13. $\frac{(n-1)(x+n)}{x+n-1} = n-(n+2)\frac{\underline{|1}}{n}\cdot\frac{x}{x+n+1}+(n+4)\frac{\underline{|2}}{n(n+1)}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}-\&c$

14. $0 = n-(n+2)\frac{n}{\underline{|1}}\cdot\frac{x}{x+n+1}+(n+4)\frac{n(n+1)}{\underline{|2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}-\&c$

15. $\frac{x+n}{2x+n} = 1-\frac{x}{x+n+1}+\frac{x(x-1)}{(x+n+1)(x+n+2)}-\&c$

16. $\frac{(n-1)(x+n)}{2x+n-1} = n-(n+2)\frac{x}{x+n+1}+(n+4)\frac{x(x-1)}{(x+n+1)(x+n+2)}$

17. $x+n = n+(n+2)\frac{x}{x+n+1}+(n+4)\frac{x(x-1)}{(x+n+1)(x+n+2)}+\&c$

18. $\frac{\underline{|x+n}\,\underline{|x-n}}{(\underline{|x})^2}\cdot\frac{\sin\pi n}{\pi} = n-(n+2)\frac{n^3}{(\underline{|1})^3}\cdot\frac{x}{x+n+1}+(n+4)\frac{n^3(n+1)^3}{(\underline{|2})^3}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}+\&c$

19. $\frac{\underline{|x+n}}{\underline{|x}\,\underline{|n}}\cdot\frac{\underline{|\frac{n}{2}}\,\underline{|x-\frac{n}{2}}}{\underline{|-\frac{n}{2}}\,\underline{|x+\frac{n}{2}}} = 1-\frac{n^2}{(\underline{|1})^2}\cdot\frac{x}{x+n+1}+\frac{n^2(n+1)^2}{(\underline{|2})^2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}-\&c$

20. $n\cdot\frac{\underline{|x-\frac{n+1}{2}}}{\underline{|x}\,\underline{|-\frac{n+1}{2}}}\cdot\frac{\underline{|x+n}\,\underline{|\frac{n-1}{2}}}{\underline{|x+\frac{n-1}{2}}\,\underline{|n}} = n-(n+2)\frac{n^2}{(\underline{|1})^2}\cdot\frac{x}{x+n+1}$
$+(n+4)\frac{n^2(n+1)^2}{(\underline{|2})^2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}$

$$\frac{1}{a+\frac{x^2}{a}} - \frac{2a}{\lfloor 1}\cdot\frac{1}{a+1+\frac{x^2}{a+1}} + \frac{2a(2a+1)}{\lfloor 2}\cdot\frac{1}{a+2+\frac{x^2}{a+2}} - \&c$$

$$= \frac{2\left(\lfloor a-1\right)^2 / \lfloor 2a-1}{\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+2}\right)^2\right\}\&c}$$

$$\int_0^\infty \frac{\phi(xi)+\phi(-xi)}{1+x^2}\,dx = \pi\,\phi(1).$$

If $\int_0^\infty \phi(x)\sin 2nx\,dx = \psi(n)$

then $\alpha\left\{\phi(\alpha)\sin 2n\alpha + \phi(2\alpha)\sin 4n\alpha + \&c\right\}$

$= \psi(n) - \psi(\beta-n) + \psi(\beta+n) - \psi(2\beta-n) + \&c$

with $\alpha\beta = \pi$ & n lying between 0 & β.

$(n-1)$ th $= \sqrt[n]{\alpha\beta} + \sqrt[n]{(1-\alpha)(1-\beta)} = 1$

then $(n-1)^2$ th $= \left(\sqrt[n]{\alpha} - \sqrt[n]{\beta}\right)^n + \left(\sqrt[n]{1-\beta} - \sqrt[n]{1-\alpha}\right)^n$

$= \left\{\sqrt[n]{\alpha(1-\beta)} - \sqrt[n]{\beta(1-\alpha)}\right\}^n.$

21. $\dfrac{n\lfloor\underline{x+n}}{\lfloor\underline{x}\,\lfloor\underline{n}} = n + (n+2)\dfrac{n^2}{(\lfloor\underline{1})^2}\cdot\dfrac{x}{x+n+1} + (n+4)\dfrac{n^2(n+1)^2}{(\lfloor\underline{2})^2}\dfrac{x(x-1)}{(x+n+1)(x+n+2)}$
$+ \&c$

22. $\dfrac{\lfloor\underline{x}\,\lfloor\underline{x+n}\,(\lfloor\underline{\tfrac{n}{2}})^2}{n\lfloor\underline{n}\,(\lfloor\underline{x+\tfrac{n}{2}})^2} = \dfrac{1}{n} - \dfrac{n}{\lfloor\underline{1}}\cdot\dfrac{x}{x+n+1}\cdot\dfrac{1}{n+2}$
$+ \dfrac{n(n+1)}{\lfloor\underline{2}}\cdot\dfrac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\dfrac{1}{n+4} - \&c$

23. $\dfrac{\lfloor\underline{2x}\,\lfloor\underline{x+n}}{\lfloor\underline{x}\,\lfloor\underline{2x+n}} = 1 - \dfrac{n}{\lfloor\underline{1}}\cdot\dfrac{x}{x+n+1} + \dfrac{n(n+1)}{\lfloor\underline{2}}\cdot\dfrac{x(x-1)}{(x+n+1)(x+n+2)} - \&c$

24. $\dfrac{\lfloor\underline{x+n}\,\lfloor\underline{\tfrac{n}{2}}}{\lfloor\underline{n}\,\lfloor\underline{x+\tfrac{n}{2}}} = 1 + \dfrac{n}{\lfloor\underline{1}}\cdot\dfrac{x}{x+n+1} + \dfrac{n(n+1)}{\lfloor\underline{2}}\cdot\dfrac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$

25. $\dfrac{\lfloor\underline{x+n}\,\lfloor\underline{\tfrac{n-1}{2}}}{\lfloor\underline{n-1}\,\lfloor\underline{x+\tfrac{n-1}{2}}} = n + (n+2)\dfrac{n}{\lfloor\underline{1}}\cdot\dfrac{x}{x+n+1} +$
$(n+4)\dfrac{n(n+1)}{\lfloor\underline{2}}\cdot\dfrac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$

26. $\dfrac{(n-1)^2}{n-2} = n + (n+2)\dfrac{1^2}{n^2} + (n+4)\dfrac{1^2 \cdot 2^2}{n^2(n+1)^2} + \&c$

27. $\dfrac{n-1}{n-2} = 1 + \dfrac{1}{n} + \dfrac{1 \cdot 2}{n(n+1)} + \dfrac{1 \cdot 2 \cdot 3}{n(n+1)(n+2)} + \&c$

28. $\dfrac{(n-1)^2}{n-3} = n + (n+2)\dfrac{1}{n} + (n+4)\dfrac{1 \cdot 2}{n(n+1)} + \&c$

29. $n-1 = n - (n+2)\dfrac{1}{n} + (n+4)\dfrac{1 \cdot 2}{n(n+1)} - \&c$

30. $\dfrac{(\lfloor\underline{n})^2}{\lfloor\underline{2n}}\cdot\dfrac{\sin\pi n\,\tan\pi n}{\pi^2 n} = n + (n+2)\dfrac{n^4}{(\lfloor\underline{1})^4} + (n+4)\dfrac{n^4(n+1)^4}{(\lfloor\underline{2})^4} + \&c$

31. $\dfrac{6(\lfloor\underline{\tfrac{n}{2}})^3\sin\pi n\,\sin\tfrac{\pi n}{2}}{\pi^2 n^2(1+2\cos\pi n)\lfloor\underline{\tfrac{3n}{2}}} = 1 + \dfrac{n^3}{(\lfloor\underline{1})^3} + \dfrac{n^3(n+1)^3}{(\lfloor\underline{2})^3} + \&c$

32. $\dfrac{\lfloor\underline{\tfrac{n-1}{2}}\,\lfloor\underline{-\tfrac{3n+1}{2}}}{(\lfloor\underline{-\tfrac{n+1}{2}})^2}\cdot\dfrac{\sin\pi n}{\pi} = n + (n+2)\dfrac{n^3}{(\lfloor\underline{1})^3} + (n+4)\dfrac{n^3(n+1)^3}{(\lfloor\underline{2})^3} + \&c$

33. $\dfrac{\sin\pi n}{\pi} = n - (n+2)\dfrac{n^3}{(\lfloor\underline{1})^3} + (n+4)\dfrac{n^3(n+1)^3}{(\lfloor\underline{2})^3} - \&c$

34. $\dfrac{(\lfloor\underline{\tfrac{n}{2}})^4}{(\lfloor\underline{n})^2}\cdot\dfrac{2\tan\tfrac{\pi n}{2}}{\pi n^2} = \dfrac{1}{n} + \dfrac{1}{n+2}\cdot\dfrac{n^2}{(\lfloor\underline{1})^2} + \dfrac{1}{n+4}\cdot\dfrac{n^2(n+1)^2}{(\lfloor\underline{2})^2} + \&c$

$$\int_0^\infty \frac{\cos 2nx\, dx}{\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+2}\right)^2\right\}\infty} = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(a-\frac{1}{2})}{\Gamma(a-1)}\operatorname{sech}^{2a} n.$$

$$\int_0^\infty \frac{1}{\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\infty}\cdot\frac{dx}{\left\{1+\left(\frac{x}{b}\right)^2\right\}\left\{1+\left(\frac{x}{b+1}\right)^2\right\}\infty}$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(a-\frac{1}{2})\,\Gamma(b-\frac{1}{2})\,\Gamma(a+b-1)}{\Gamma(a-1)\,\Gamma(b-1)\,\Gamma(a+b-\frac{1}{2})}.$$

$$\int_0^\infty \frac{dx}{\left(1+\frac{x^2}{a^2}\right)\left(1+\frac{x^2}{b^2}\right)\left(1+\frac{x^2}{c^2}\right)\left(1+\frac{x^2}{d^2}\right)}$$

$$= \frac{\pi}{6}\cdot\frac{abcd\left\{(a+b+c+d)^3-(a^3+b^3+c^3+d^3)\right\}}{(a+b)(b+c)(c+a)(a+d)(b+d)(c+d)}$$

If $\alpha, \beta, \gamma, \& \delta$ are the roots of the equation

$x^4 - px^3 + qx^2 - rx + s = 0$, then

$$\int_0^\infty \frac{dx}{\left(1+\frac{x^2}{\alpha^2}\right)\left(1+\frac{x^2}{\beta^2}\right)\left(1+\frac{x^2}{\gamma^2}\right)\left(1+\frac{x^2}{\delta^2}\right)} = \frac{\pi}{2}\cdot\frac{s}{r-\frac{ps}{q-\frac{r}{p}}}.$$

If $\int_0^\infty \phi(a,x)\cos nx\, dx = \psi(a,n)$, then

$$\frac{\pi}{2}\int_0^\infty \phi(a,x)\,\phi(b,x)\,dx = \int_0^\infty \psi(a,x)\,\psi(b,x)\,dx$$

$$35.\ \frac{\pi\left(\lfloor\frac{x}{2}\right)^2}{2x\lfloor x\,\sin\frac{\pi x}{2}} = \frac{1}{x^2} + \frac{1}{(x+2)^2}\cdot\frac{x}{\lfloor 1} + \frac{1}{(x+4)^2}\cdot\frac{x(x+1)}{\lfloor 2} + \&c$$

$$36.\ \Sigma\frac{1}{x+n} - \Sigma\frac{1}{n} = \left(1+\frac{1}{n+1}\right)\frac{x}{x+n+1} + \left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$$

$$37.\ \Sigma\frac{1}{x} + \Sigma\frac{1}{n} - \Sigma\frac{1}{x+n} = \left(1+\frac{1}{n+1}\right)\frac{n}{\lfloor 1}\cdot\frac{x}{x+n+1} - \left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{n(n+1)}{\lfloor 2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$$

$$38.\ 2n^2\left\{\frac{1}{n^3} + \frac{1}{(n+1)^3} + \frac{1}{(n+2)^3} + \&c\right\}$$
$$= \left(1+\frac{1}{n}\right) + \left(\frac{1}{2}+\frac{1}{n+1}\right)\left(\frac{1}{n+1}\right)^2 + \left(\frac{1}{3}+\frac{1}{n+2}\right)\left(\frac{1}{n+1}\cdot\frac{2}{n+2}\right)^2 + \&c$$

$$39.\ \left\{\frac{1}{\left(1+\frac{n}{2}\right)^2} + \frac{1}{\left(2+\frac{n}{2}\right)^2} + \&c\right\} - \left\{\frac{1}{(1+n)^2} + \frac{1}{(2+n)^2} + \&c\right\}$$
$$= \left(1-\frac{1}{n+1}\right)\frac{1}{n+1} + \left(\frac{1}{2}-\frac{1}{n+2}\right)\frac{\lfloor 1}{(n+1)(n+2)} + \&c$$

$$40.\ \Sigma\frac{1}{n} + \Sigma\frac{1}{-n} = \left(1+\frac{1}{n+1}\right)\frac{n^2}{(\lfloor 1)^2} + \left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{n^2(n+1)^2}{(\lfloor 2)^2} + \&c$$

$$\text{Ex. } 1.\ \frac{(\lfloor x)^3\lfloor 3x-2}{(\lfloor 2x-1)^3} = 1 - 3\left(\frac{x-1}{x+1}\right)^3 + 5\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^3 - \&c$$

$$2.\ \frac{x^2}{2x-1} = 1 + 3\left(\frac{x-1}{x+1}\right)^2 + 5\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^2 + \&c$$

$$3.\ \frac{(\lfloor x)^4\lfloor 4x}{(\lfloor 2x)^4}\cdot\frac{2x}{4x-1} = 1 + \left(\frac{x-1}{x+1}\right)^2 + \left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^2 + \&c$$

$$4.\ \frac{(\lfloor x)^2}{\lfloor 2x-1} = 1 - 3\left(\frac{x-1}{x+1}\right)^2 + 5\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^2 - \&c$$

$$5.\ x = 1 + 3\,\frac{x-1}{x+1} + 5\,\frac{x-1}{x+1}\cdot\frac{x-2}{x+2} + \&c$$

$$6.\ \frac{2^{2x-1}(\lfloor x)^2}{\lfloor 2x} = 1 + \frac{x-1}{x+1} + \frac{x-1}{x+1}\cdot\frac{x-2}{x+2} + \&c$$

$$7.\ \frac{x}{2x-1} = 1 - \frac{x-1}{x+1} + \frac{x-1}{x+1}\cdot\frac{x-2}{x+2} - \&c$$

$$8.\ 0 = 1 - 3\,\frac{x-1}{x+1} + 5\,\frac{x-1}{x+1}\cdot\frac{x-2}{x+2} - \&c$$

$$9.\ \frac{1}{2}\Sigma\frac{1}{x-1} + \frac{(2^{x-1}\lfloor x-1)^2}{\lfloor 2x-1} = 1 + \frac{1}{2}\cdot\frac{x-1}{x+1} + \frac{1}{3}\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}$$

$$10.\ \Sigma\frac{1}{2x} - \Sigma\frac{1}{x} + \frac{1}{2x} = 1 - \frac{1}{2}\cdot\frac{x-1}{x+1} + \frac{1}{3}\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}$$

$$\int_0^\infty x^{n-1}\left\{\phi(0) - \frac{x}{\lfloor\underline{1}}\phi(1) + \frac{x^2}{\lfloor\underline{2}}\phi(2) - \frac{x^3}{\lfloor\underline{3}}\phi(3) + \&c\right\} dx$$

$$= \lfloor\underline{n-1}\;\phi(-n).$$

$$\lfloor\underline{n}\int_0^\infty \frac{\lfloor\underline{x+p-1}}{x^m\,\lfloor\underline{x+p+n}}\,dx$$

$$= \frac{\pi}{\sin\pi m}\left\{\frac{1}{p^m} - \frac{n}{\lfloor\underline{1}}\cdot\frac{1}{(p+1)^m} + \frac{n(n-1)}{\lfloor\underline{2}}\cdot\frac{1}{(p+2)^m} - \&c\right\}$$

$$\int_0^\infty \frac{1}{x^{n+1}}\cdot\frac{1-px}{1+x}\cdot\frac{1-prx}{1+rx}\cdot\frac{1-pr^2x}{1+r^2x}\cdot\frac{1-pr^3x}{1+r^3x}\,\&c\;dx$$

$$= \cdot\frac{1+p}{1+pr^n}\cdot\frac{1+pr}{1+pr^{n+1}}\cdot\frac{1+pr^2}{1+pr^{n+2}}\,\&c$$

$$\times\frac{1-r^{n+1}}{1-r}\cdot\frac{1-r^{n+2}}{1-r^2}\cdot\frac{1-r^{n+3}}{1-r^3}\,\&c \times -\frac{\pi}{\sin\pi n}.$$

$$\int_0^\infty \frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+2}\right)^2}\,\&c\;dx$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\lfloor\underline{a-\frac{1}{2}}\,\lfloor\underline{b}}{\lfloor\underline{a-1}\,\lfloor\underline{b-\frac{1}{2}}}\cdot\frac{\lfloor\underline{b-a-\frac{1}{2}}}{\lfloor\underline{b-a}}.$$

Subtract.

$\frac{[illegible](\lfloor x)^2}{4x(\lfloor 2x)^2} = 1 - \frac{[illegible]}{3}\cdot\frac{x-1}{x+1} + [illegible]\frac{[illegible]}{x+1}\cdot\frac{[illegible]}{x+2}$ &c

12. $\frac{1}{2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{x^2}\right) + x\left(\frac{1}{x+1}+\frac{1}{x+2}+\cdots\frac{1}{x+x}\right)$

$= 1 - \frac{1}{2^2}\cdot\frac{x-1}{x+1} + \frac{1}{3^2}\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2} - \frac{1}{4^2}\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\cdot\frac{x-3}{x+3} +$ &c

13. $x(4x-3) = 1^3 + 3^3\frac{x-1}{x+1} + 5^3\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2} +$ &c

14. $\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1\cdot3}{2\cdot4}\right)^3 - 13\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^3 +$ &c

15. $\frac{\pi^2}{8}\left\{1 + 9\left(\frac{1}{4}\right)^4 + 17\left(\frac{1\cdot5}{4\cdot8}\right)^4 + 25\left(\frac{1\cdot5\cdot9}{4\cdot8\cdot12}\right)^4 + \text{\&c}\right\}$

$= 1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1\cdot3}{2\cdot4}\right)^3 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^3 +$ &c

16. $1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1\cdot3}{2\cdot4}\right)^3 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^3 + \text{\&c} = 2\left\{1 \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot3}{2\cdot4}\right)^2 \text{\&c}\right\}^2$

17. $1 + \frac{1}{5}\left(\frac{1}{2}\right)^2 + \frac{1}{9}\left(\frac{1\cdot3}{2\cdot4}\right)^2 + \frac{1}{13}\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 +$ &c

$= \frac{\pi}{2}\left\{1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot3}{2\cdot4}\right)^2 - \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 + \text{\&c}\right\}$

18. $\frac{1}{1^2} + \left(\frac{1}{2}\right)^2\frac{1}{5^2} + \left(\frac{1\cdot3}{2\cdot4}\right)^2\frac{1}{9^2} + \text{\&c} = \frac{\pi^3}{32}\left\{1 + 9\left(\frac{1}{4}\right)^4 + 17\left(\frac{1\cdot5}{4\cdot8}\right)^4 + \text{\&c}\right\}$

41. $\frac{\lfloor n\;\lfloor x+y+n}{\lfloor x+n\;\lfloor y+n} = 1 + \frac{x}{\lfloor 1}\cdot\frac{y}{n+1} + \frac{x(x-1)}{\lfloor 2}\cdot\frac{y(y-1)}{(n+1)(n+2)} +$ &c

Sol. Write $-n+m$ for z in XII 5 and make m infinite or equate the coeff[ts] of A^x in $(1+A)^{y+n}\left(1+\frac{1}{A}\right)^x = \frac{(1+A)^{x+y+n}}{A^x}$

42. $\Sigma\frac{1}{a+1} - \Sigma\frac{1}{a-b+1} = \frac{b}{a} + \frac{b(b+1)}{a(a+1)}\cdot\frac{1}{2} + \frac{b(b+1)(b+2)}{a(a+1)(a+2)}\cdot\frac{1}{3} +$ &c

Sol. Subtract 1 from both sides in XII 41, divide by y and put $y=0$

43. $\frac{\lfloor n\;\lfloor x}{n\;\lfloor x+n} = \frac{1}{n} - \frac{x}{\lfloor 1}\cdot\frac{1}{n+1} + \frac{x(x-1)}{\lfloor 2}\cdot\frac{1}{n+2} -$ &c

Sol. Write $-n$ for y in XII 41.

[illegible] $\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} + [illegible]$ &c

$= \frac{1}{n+1} + \frac{\lfloor 1}{2(n+1)(n+2)} + \frac{\lfloor 2}{3(n+1)[illegible]}$

$$\int_0^\infty \frac{\lfloor\!\underline{x+\alpha}}{\lfloor\!\underline{x+\beta}} \left(\Sigma \frac{1}{x+\beta} - \Sigma \frac{1}{x+\alpha}\right) dx$$

$$= \frac{\lfloor\!\underline{\alpha}}{\lfloor\!\underline{\beta}}.$$ Integration very simple.

$$\int_0^{\frac{\pi}{2}} \cos^m x \cos nx \, dx = \frac{\pi \lfloor\!\underline{m}}{2^{m+1} \lfloor\!\underline{\frac{m+n}{2}} \lfloor\!\underline{\frac{m-n}{2}}}.$$

$$\int_0^\infty \frac{\sin^n x}{x^p} dx = \frac{1}{\lfloor\!\underline{p-1}} \int_0^\infty \int_0^\infty z^{p-1} e^{-zx} \sin^n x \, dz \, dx$$

$$\int_0^\infty e^{-ax} \sin^{2n+1} x \, dx = \frac{\lfloor\!\underline{2n+1}}{(a^2+1^2)(a^2+3^2) \ldots (a^2+\overline{2n+1}^2)}$$

$$\int_0^\infty e^{-ax} \sin^{2n} x \, dx = \frac{\lfloor\!\underline{2n}}{a(a^2+2^2)(a^2+4^2) \ldots (a^2+\overline{2n}^2)}$$

change x to xi then we get $e^x - e^{-x}$

2. $\frac{\pi}{\sin \pi n} = \frac{1}{n} + \frac{n}{\underline{|1}}\cdot\frac{1}{n+1} + \frac{n(n+1)}{\underline{|2}}\cdot\frac{1}{n+2} + \&c$

3. $\frac{(2^x \underline{|x})^2}{x\,\underline{|2x}} = \frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x+1} + \frac{1.3}{2.4}\cdot\frac{1}{x+2} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{x+3} + \&c$

4. $\frac{\pi}{\tan\pi x}\cdot\frac{\underline{|2x}}{(2^x\underline{|x})^2(1-2x)} = 1 + \frac{x}{\underline{|1}}\cdot\frac{1}{3} + \frac{x(x+1)}{\underline{|2}}\cdot\frac{1}{5} + \&c$

5. $\frac{\underline{|x}\,\underline{|n-1}}{\underline{|x+n}}\left(\Sigma\frac{1}{x+n} - \Sigma\frac{1}{n-1}\right) = \frac{1}{n^2} - \frac{x}{\underline{|1}}\cdot\frac{1}{(n+1)^2} + \frac{x(x-1)}{\underline{|2}}\cdot\frac{1}{(n+2)^2}$

6. $\frac{(2^x\underline{|x})^2}{x\,\underline{|2x}}\left(2\Sigma\frac{1}{2x} - 2\Sigma\frac{1}{x} + \frac{1}{x} - 2\log_e 2\right)$

$= \frac{1}{x^2} + \frac{1}{2}\cdot\frac{1}{(x+1)^2} + \frac{1.3}{2.4}\cdot\frac{1}{(x+2)^2} + \&c$

7. $\frac{\pi}{\tan\pi x}\cdot\frac{\underline{|2x}}{(1-2x)(2^x\underline{|x})^2}\left(\Sigma\frac{1}{2x} - \frac{1}{2}\Sigma\frac{1}{x} + \frac{1}{1-2x} - \frac{\pi}{2}\tan\pi x\right)$

$= \frac{1}{1^2} + \frac{x}{\underline{|1}}\cdot\frac{1}{3^2} + \frac{x(x+1)}{\underline{|2}}\cdot\frac{1}{5^2} + \&c$

8. $-\frac{\pi}{\sin\pi n}\Sigma\frac{1}{n-1} = \frac{1}{n^2} + \frac{n}{\underline{|1}}\cdot\frac{1}{(n+1)^2} + \frac{n(n+1)}{\underline{|2}}\cdot\frac{1}{(n+2)^2} + \&c$

44. $a^n = \{a^n - (b+1)^n\} + \{(a+1)^n - (b+2)^n\}\left(\frac{b+1}{a+1}\right)^n +$

$\{(a+2)^n - (b+3)^n\}\left(\frac{b+1}{a+1}\right)^n\left(\frac{b+2}{a+2}\right)^n + \&c$

Sol. $a^n = \{a^n - (b+1)^n\} + (b+1)^n = \{a^n - (b+1)^n\} + (a+1)^n\left(\frac{b+1}{a+1}\right)^n$

$= \&c\ \&c$

Cor 1. $\frac{b}{a-b-1} = \frac{b}{a} + \frac{b(b+1)}{a(a+1)} + \frac{b(b+1)(b+2)}{a(a+1)(a+2)} + \&c$

2. $\frac{b^2}{a-b-1} = (a+b+1)\frac{b^2}{a^2} + (a+b+3)\frac{b^2(b+1)^2}{a^2(a+1)^2} + \&c$

45. If $e^{A_1 x + A_2\frac{x^2}{2} + A_3\frac{x^3}{3} + \&c}$

$= P_0 + P_1 x + P_2 x^2 + \&c$, then

$P_n = P_{n-1}A_1 + P_{n-2}A_2 + P_{n-3}A_3 + \&c$ to n terms and [illegible]

Sol. Take logarithms of both sides and then differentiate them.

If $S_2 = a_1^2 + a_2^2 + \cdots + a_n^2$ and

$\frac{ab}{cd} = k$. Then

$$f(a,b)f(c,d) + f(-a,-b)f(-c,-d)\}$$

$$= f(ac, bd) + ad\, f\left(ack, \frac{bd}{k}\right) + bc\, f\left(bdk, \frac{ac}{k}\right)$$

$$+ (ad)^3 bc\, f\left(ack^2, \frac{bd}{k^2}\right) + (bc)^3 ad\, f\left(bdk^2, \frac{ac}{k^2}\right)$$

$$+ (ad)^6 (bc)^3 f\left(ack^3, \frac{bd}{k^3}\right) + (bc)^6 (ad)^3 f\left(bdk^3, \frac{ac}{k^3}\right)$$

$$+ \&c$$

$$\frac{1}{2}\{f(a,b)f(c,d) - f(-a,-b)f(-c,-d)\}$$

$$= a\, f\left(\frac{c}{a}, \frac{a}{c}\cdot abcd\right) + d\, f\left(\frac{b}{d}, \frac{d}{b}\cdot abcd\right)$$

$$+ a^3 bc\, f\left(\frac{c}{ak}, \frac{ak}{c}\cdot abcd\right) + d^3 bc\, f\left(\frac{bk}{d}, \frac{d}{bk}\cdot abcd\right)$$

$$+ a^6 d (bc)^3 f\left(\frac{c}{ak^2}, \frac{ak^2}{c}\cdot abcd\right) + ad^6 (bc)^3 f\left(\frac{bk^2}{d}, \frac{d}{bk^2}\cdot abcd\right)$$

$$+ \&c \quad \&c \quad \&c$$

of the product of $a_1, a_2, a_3 \dots a_n$ taken r at a time

then $rP_r = P_{r-1}S_1 - P_{r-2}S_2 + P_{r-3}S_3 - P_{r-4}S_4 + \&c$ and $P_0 = 1$

46. $\frac{1}{n^{r+1}} - \frac{x}{\lfloor 1} \cdot \frac{1}{(n+1)^{r+1}} + \frac{x(x-1)}{\lfloor 2} \cdot \frac{1}{(n+2)^{r+1}} - \&c$

$= \frac{\lfloor x \lfloor n}{n \lfloor x+n} \phi(r)$ where $\phi(0) = 1$ and

$r\phi(r) = S_1\phi(r-1) + S_2\phi(r-2) + S_3\phi(r-3) + \&c$ to r terms

where $S_r = \frac{1}{n^r} - \frac{1}{(x+n+1)^r} + \frac{1}{(n+1)^r} - \frac{1}{(x+n+2)^r} + \&c$

Cor 1. $1 + \frac{1}{2} \cdot \frac{1}{3^{r+1}} + \frac{1.3}{2.4} \cdot \frac{1}{5^{r+1}} + \frac{1.3.5}{2.4.6} \cdot \frac{1}{7^{r+1}} + \&c = \frac{\pi}{2}\phi(r)$

where $\phi(0) = 1$ and $r\phi(r) = S_1\phi(r-1) + S_2\phi(r-2) + S_3\phi(r-3) + \&c$

to r terms where $S_r = \frac{1}{1^r} - \frac{1}{2^r} + \frac{1}{3^r} - \frac{1}{4^r} + \&c$

Cor 2. $\frac{1}{2^{r+1}} + \frac{1}{2} \cdot \frac{1}{4^{r+1}} + \frac{1.3}{2.4} \cdot \frac{1}{6^{r+1}} + \&c = \phi(r)$ where

$\phi(0) = 1$ and $r\phi(r) = S_1\phi(r-1) + S_2\phi(r-2) + \&c$ where

$S_r = \frac{1}{2^r} - \frac{1}{3^r} + \frac{1}{4^r} - \frac{1}{5^r} + \&c.$

Sol. Write $n - p$ for n in XII 43 and equate the coeffts of p^r

Ex. $1 + \frac{1}{2} \cdot \frac{1}{3^3} + \frac{1.3}{2.4} \cdot \frac{1}{5^3} + \&c = \frac{\pi^3}{48} + \frac{\pi}{4}(\log 2)^2$.

47. $\frac{1}{(x+1)^n} + \frac{1 + \frac{1}{2}}{(x+2)^n} + \frac{1 + \frac{1}{2} + \frac{1}{3}}{(x+3)^n} + \frac{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}}{(x+4)^n} + \&c = \frac{n}{2}S_{n+1}$

$-(S_1 S_n + S_2 S_{n-1} + S_3 S_{n-2} + \&c$ the last term being $S_{\frac{n}{2}} S_{\frac{n+2}{2}}$

or $\frac{1}{2} S_{\frac{n+1}{2}} S_{\frac{n+1}{2}}$ according as n is even or odd) where

$S_n = \frac{1}{x^n} + \frac{1}{(x+1)^n} + \frac{1}{(x+2)^n} + \&c$ and $S_1 = -C - \frac{1}{x-1}$

Sol. $1(\frac{1}{2} - \frac{1}{n+2}) + (1 + \frac{1}{2})(\frac{1}{3} - \frac{1}{n+3}) + (1 + \frac{1}{2} + \frac{1}{3})(\frac{1}{4} - \frac{1}{n+4}) + \&c$

$= \frac{1}{2}\{(1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n})^2 + (\frac{1}{1^2} + \frac{1}{2^2} + \dots \frac{1}{n^2})\}$

In this identity write $n + x - 1$ for n and equate

the coeffts of x^n ...

If $P = \dfrac{l}{x+}\,\dfrac{1^2-n^2}{x+}\,\dfrac{2^2-l^2}{x+}\,\dfrac{3^2-n^2}{x+}\,\dfrac{4^2-l^2}{x+\&c}$

then $$\frac{1-P}{1+P} = \frac{\left|\underline{\frac{x+l+n-3}{4}}\right.\ \left|\underline{\frac{x+l-n-3}{4}}\right.\ \left|\underline{\frac{x-l+n-1}{4}}\right.\ \left|\underline{\frac{x-l-n-1}{4}}\right.}{\left|\underline{\frac{x-l+n-3}{4}}\right.\ \left|\underline{\frac{x-l-n-3}{4}}\right.\ \left|\underline{\frac{x+l+n-1}{4}}\right.\ \left|\underline{\frac{x+l-n-1}{4}}\right.}$$

$$\frac{\left|\underline{\frac{x+l+n-3}{4}}\right.\ \left|\underline{\frac{x-l+n-3}{4}}\right.\ \left|\underline{\frac{x+l-n-3}{4}}\right.\ \left|\underline{\frac{x-l-n-3}{4}}\right.}{\left|\underline{\frac{x+l+n-1}{4}}\right.\ \left|\underline{\frac{x-l+n-1}{4}}\right.\ \left|\underline{\frac{x+l-n-1}{4}}\right.\ \left|\underline{\frac{x-l-n-1}{4}}\right.}$$

$$= \frac{8}{\frac{x^2-l^2+n^2-1}{2} +}\,\frac{1^2-n^2}{1+}\,\frac{1^2-l^2}{x^2-1+}\,\frac{3^2-n^2}{1+}\,\frac{3^2-l^2}{x^2-1+\&c}$$

$$\left\{\frac{1}{(x-n+1)^2} + \frac{1}{(x-n+3)^2} + \frac{1}{(x-n+5)^2} + \&c\right\}$$

$$-\left\{\frac{1}{(x+n+1)^2} + \frac{1}{(x+n+3)^2} + \frac{1}{(x+n+5)^2} + \&c\right\}$$

$$= \frac{n}{x^2-1+n^2+}\,\frac{2(1^2-n^2)}{1+}\,\frac{2}{3(x^2-1)+n^2+}\,\frac{4(2^2-n^2)}{1+\&c}$$

$$= \frac{n}{x^2-n^2+1-}\,\frac{4(1^2-n^2)}{3(x^2-n^2+5)-\&c}$$

CHAPTER XIII

1. If $A_0 - nA_1 + \frac{n(n-1)}{|\underline{2}}A_2 - \frac{n(n-1)(n-2)}{|\underline{3}}A_3 + \&c = P_n$, then

$$P_0 - nP_1 + \frac{n(n-1)}{|\underline{2}}P_2 - \frac{n(n-1)(n-2)}{|\underline{3}}P_3 + \&c = A_n.$$

2. $\frac{A_0}{x^n} + \frac{n}{|\underline{1}}\cdot\frac{A_1}{x^{n+1}} + \frac{n(n+1)}{|\underline{2}}\cdot\frac{A_2}{x^{n+2}} + \&c$

$$= \frac{A_0}{(x+h)^n} + \frac{n}{|\underline{1}}\cdot\frac{A_1 + hA_0}{(x+h)^{n+1}} + \frac{n(n+1)}{|\underline{2}}\cdot\frac{A_2 + 2hA_1 + h^2A_0}{(x+h)^{n+2}} + \&c$$

3. If $\frac{A_0}{x^n} + \frac{n}{|\underline{1}}\cdot\frac{A_1}{x^{n+1}} + \frac{n(n+1)}{|\underline{2}}\cdot\frac{A_2}{x^{n+2}} + \&c$

$= \frac{A_0}{(x-1)^n} - \frac{n}{|\underline{1}}\cdot\frac{A_1}{(x-1)^{n+1}} + \frac{n(n+1)}{|\underline{2}}\cdot\frac{A_2}{(x-1)^{n+2}} - \&c$, then

(a) $$e^x = \frac{A_0 + \frac{x}{|\underline{1}}A_1 + \frac{x^2}{|\underline{2}}A_2 + \frac{x^3}{|\underline{3}}A_3 + \&c}{A_0 - \frac{x}{|\underline{1}}A_1 + \frac{x^2}{|\underline{2}}A_2 - \frac{x^3}{|\underline{3}}A_3 + \&c}$$

Sol. Multiply both sides in XIII 3 by x^n and make n and in-
-finite such that $\frac{n}{x} = y$

(b). $$\frac{1}{\{\phi(x)\}^n}\left[A_0 + A_1\frac{n}{|\underline{1}}\left\{\frac{\phi(x)-\phi(-x)}{\phi(x)}\right\} + A_2\frac{n(n+1)}{|\underline{2}}\left\{\frac{\phi(x)-\phi(-x)}{\phi(x)}\right\}^2 + \&c\right]$$

is always an even function of x whatever be $\phi(x)$

Sol. Writing in XIII 3 $\frac{\phi(x)}{\phi(x)-\phi(-x)}$ for x, we have

$$\frac{A_0}{\{\phi(x)\}^n} + A_1\frac{n}{|\underline{1}}\frac{\phi(x)-\phi(-x)}{\{\phi(x)\}^{n+1}} + A_2\frac{n(n+1)}{|\underline{2}}\frac{\{\phi(x)-\phi(-x)\}^2}{\{\phi(x)\}^{n+2}} + \&c$$

$$= \frac{A_0}{\{\phi(-x)\}^n} + A_1\frac{n}{|\underline{1}}\cdot\frac{\phi(-x)-\phi(x)}{\{\phi(-x)\}^{n+1}} + A_2\frac{n(n+1)}{|\underline{2}}\cdot\frac{\{\phi(-x)-\phi(x)\}^2}{\{\phi(-x)\}^{n+2}} + \&c$$

$\therefore$ Each of these is an even function of x.

(c) If n is even, the value of A_{n+1} depends upon the value of A_n.
But we may give for A_n any value we choose.

$$\frac{A_{n-1}}{2} = \frac{n-1}{|\underline{2}}(2^2-1)B_2A_{n-2} - \frac{(n-1)(n-2)(n-3)}{|\underline{4}}(2^4-1)B_4A_{n-4} +$$

$$\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{|\underline{6}}(2^6-1)B_6A_{n-6} - \&c$$

$$\phi(0) + \frac{m}{n}\cdot\frac{\phi'(0)}{\lfloor 1} + \frac{m(m+1)}{n(n+1)}\frac{\phi''(0)}{\lfloor 2} + \&c\,.$$

$$= \phi(1) + \frac{m-n}{n}\,\frac{\phi'(1)}{\lfloor 1} + \frac{(m-n)(m-n-1)}{n(n+1)}\cdot\frac{\phi''(1)}{\lfloor 2} + \&c$$

$$V = \cfrac{2lmn}{y+p-2ml^2 + \cfrac{2(1-m)(1^2-n^2)}{1 + \cfrac{2(1+m)(1^2-l^2)}{3y+p + \cfrac{2(2-m)(2^2-n^2)}{1 + \cfrac{2(2+m)(2^2-l^2)}{5y+p + \&c\ \&c}}}}}$$

where $y = x^2 - (1-m)^2$ & $p = (n^2-l^2)(1-2m)$.

If $\phi(x,y) = x + \cfrac{(1+y)^2+n}{2x + \cfrac{(3+y)^2+n}{2x + \cfrac{(5+y)^2+n}{2x + \&c}}}$

then $\phi(x,y) = \phi(y,x)$.

Cor.

Sol. We have from XIII 3 (a),

$$A_0 + \frac{x}{\underline{|1}} A_1 + \frac{x^2}{\underline{|2}} A_2 + \&c = e^x \left(A_0 - \frac{x}{\underline{|1}} A_1 + \frac{x^2}{\underline{|2}} A_2 - \&c\right).$$

i.e. $(e^{-x}+1)\left(A_0 + \frac{x}{\underline{|1}} A_1 + \frac{x^2}{\underline{|2}} A_2 + \&c\right) = (e^x+1)\left(A_0 - \frac{x}{\underline{|1}} A_1 + \frac{x^2}{\underline{|2}} - \&c\right)$

$\therefore \frac{1}{e^x+1}\left(A_0 + \frac{x}{\underline{|1}} A_1 + \frac{x^2}{\underline{|2}} A_2 + \&c\right)$ is an even function of x

Since n is even the coeff^t. of x^{n-1} must be 0.

4. $$\frac{1}{x^n} + \frac{n}{\underline{|1}} \cdot \frac{m}{n} \cdot \frac{1}{x^{n+1}} + \frac{n(n+1)}{\underline{|2}} \cdot \frac{m(m+1)}{n(n+1)} \cdot \frac{1}{x^{n+2}} + \&c$$

$$= \frac{1}{(x-1)^n} + \frac{n}{\underline{|1}} \cdot \frac{m-n}{n} \cdot \frac{1}{(x-1)^{n+1}} + \frac{n(n+1)}{\underline{|2}} \cdot \frac{(m-n)(m-n-1)}{n(n+1)} \cdot \frac{1}{(x-1)^{n+2}} + \&c$$

Sol. We have by

$$\frac{\underline{|n-1}\,\underline{|m+k-1}}{\underline{|n+k-1}\,\underline{|m-1}} = 1 + \frac{k}{\underline{|1}} \cdot \frac{m-n}{n} + \frac{k(k-1)}{\underline{|2}} \cdot \frac{(m-n)(m-n-1)}{n(n+1)} + \&c$$

multiplying both sides by $\frac{\underline{|n+k-1}}{\underline{|n-1}\,\underline{|k}}$ we have

$$\frac{\underline{|n+k-1}}{\underline{|k}\,\underline{|n-1}} \cdot \frac{\underline{|n-1}\,\underline{|m+k-1}}{\underline{|n+k-1}\,\underline{|m-1}} = \frac{\underline{|n+k-1}}{\underline{|n-1}\,\underline{|k}} + \frac{n}{\underline{|1}} \cdot \frac{m-n}{n} \cdot \frac{\underline{|n+k-1}}{\underline{|n}\,\underline{|k-1}} + \&c$$

$\therefore$ The series in which L.H.S is the coeff^t. of $\frac{1}{x^{n+k}}$

= that in which R.H.S is the coeff^t. of $\frac{1}{x^{n+k}}$

5. $$e^x = \frac{1 + \frac{m}{n} \cdot \frac{x}{\underline{|1}} + \frac{m(m+1)}{n(n+1)} \cdot \frac{x^2}{\underline{|2}} + \frac{m(m+1)(m+2)}{n(n+1)(n+2)} \cdot \frac{x^3}{\underline{|3}} + \&c}{1 + \frac{m-n}{n} \cdot \frac{x}{\underline{|1}} + \frac{(m-n)(m-n-1)}{n(n+1)} \cdot \frac{x^2}{\underline{|2}} + \frac{(m-n)(m-n-1)(m-n-2)}{n(n+1)(n+2)} \cdot \frac{x^3}{\underline{|3}} + \&c}$$

Sol. multiply both sides in XIII 4 by x^n; make x & n infinite such that $\frac{n}{x} = y$.

6. $$\frac{1}{x^n} + \frac{n}{\underline{|1}} \cdot \frac{m}{2m} \cdot \frac{1}{x^{n+1}} + \frac{n(n+1)}{\underline{|2}} \cdot \frac{m(m+1)}{2m(2m+1)} \cdot \frac{1}{x^{n+2}} + \&c$$

$$= \frac{1}{(x-1)^n} - \frac{n}{\underline{|1}} \cdot \frac{m}{2m} \cdot \frac{1}{(x-1)^{n+1}} + \frac{n(n+1)}{\underline{|2}} \cdot \frac{m(m+1)}{2m(2m+1)} \cdot \frac{1}{(x-1)^{n+2}} - \&c$$

7. $$e^x = \frac{1 + \frac{m}{2m} \cdot \frac{x}{\underline{|1}} + \frac{m(m+1)}{2m(2m+1)} \cdot \frac{x^2}{\underline{|2}} + \frac{m(m+1)(m+2)}{2m(2m+1)(2m+2)} \cdot \frac{x^3}{\underline{|3}} + \&c}{1 - \frac{m}{2m} \cdot \frac{x}{\underline{|1}} + \frac{m(m+1)}{2m(2m+1)} \cdot \frac{x^2}{\underline{|2}} - \frac{m(m+1)}{\dots} + \&c}$$

Ex. 1. $$e^x = \frac{1 + \frac{1}{2} \cdot \frac{x}{\underline{|1}} + \frac{1 \cdot 3}{\dots} \cdots}{1 - \frac{1}{2} \cdot \frac{x}{\underline{|1}} + \frac{1 \cdot 3}{\dots} \cdot \frac{x^2}{\dots} - \frac{1 \cdot 3 \cdot 5}{\dots} \cdots}$$

If

$$K = \frac{\left(\frac{x+l+m+n-1}{2}\right)!\,\left(\frac{x+l-m-n-1}{2}\right)!\,\left(\frac{x+m-n-l-1}{2}\right)!\,\left(\frac{x+n-l-m-1}{2}\right)!}{\left(\frac{x-l-m-n-1}{2}\right)!\,\left(\frac{x-l+m+n-1}{2}\right)!\,\left(\frac{x-m+n+l-1}{2}\right)!\,\left(\frac{x-n+l+m-1}{2}\right)!}$$

then

$$\frac{1-K}{1+K} = \cfrac{2lmn}{x^2-l^2-m^2-n^2+1+\cfrac{4(l^2-1^2)(m^2-1^2)(n^2-1^2)}{3(x^2-l^2-m^2-n^2+5)+\&c}} \\ + \cfrac{4(l^2-2^2)(m^2-2^2)(n^2-2^2)}{5(x^2-l^2-m^2-n^2+9)+\&c} = V$$

If

$$K = \frac{\left(\frac{x+l+n-1}{4}\right)!\,\left(\frac{x+l-n-3}{4}\right)!\,\left(\frac{x-l+n-3}{4}\right)!\,\left(\frac{x-l-n-1}{4}\right)!}{\left(\frac{x-l+n-1}{4}\right)!\,\left(\frac{x-l-n-3}{4}\right)!\,\left(\frac{x+l-n-1}{4}\right)!\,\left(\frac{x+l+n-3}{4}\right)!}$$

then

$$\frac{1-K}{1+K} = \cfrac{ln}{x^2-1-l^2+\cfrac{2^2-n^2}{1+\cfrac{2^2-l^2}{x^2-1+\cfrac{4^2-n^2}{1+\cfrac{4^2-l^2}{x^2-1+\&c}}}}}$$

If $\phi(y) = \frac{1}{y+1} - \frac{1}{y+3} + \frac{1}{y+5} + \&c$

then

$$\phi(x-l-n) - \phi(x+l-n) + \phi(x+l+n) \\ - \phi(x-l+n) = \cfrac{2ln}{(x^2-1)+n^2-l^2+\cfrac{2(1^2-n^2)}{1+\cfrac{2(1^2-l^2)}{3(x^2-1)+n^2-l^2+\cfrac{4(2^2-n^2)}{1+\cfrac{4(2^2-l^2)}{\&c\ \&c}}}}}$$

2. $\sqrt{1-\frac{1}{x}} = \dfrac{1-\left(\frac{1}{2}\right)^2\frac{1}{x-1}+\left(\frac{1.3}{2.4}\right)^2\cdot\frac{1}{(x-1)^2}-\left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{(x-1)^3}+\&c}{1+\left(\frac{1}{2}\right)^2\frac{1}{x}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{x^2}+\left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{x^3}+\&c}$

3. $1+\left(\frac{1}{2}\right)^3+\left(\frac{1.3}{2.4}\right)^3+\left(\frac{1.3.5}{2.4.6}\right)^3+\left(\frac{1.3.5.7}{2.4.6.8}\right)^3+\&c$

$= 2\left\{1-\left(\frac{1}{2}\right)^2+\left(\frac{1.3}{2.4}\right)^2-\left(\frac{1.3.5}{2.4.6}\right)^2+\&c\right\}$

$= \left\{1+\left(\frac{1}{2}\right)^2\frac{1}{2}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{2^2}+\left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{2^3}+\&c\right\}^2$

8. $\frac{1}{nx^n}+\frac{m}{\underline{|1}}\cdot\frac{1}{(n+1)x^{n+1}}+\frac{m(m+1)}{\underline{|2}}\cdot\frac{1}{(n+2)x^{n+2}}+\&c$

$= \frac{1}{n(x-1)^n}+\frac{m-n-1}{\underline{|1}}\cdot\frac{1}{(n+1)(x-1)^{n+1}}+\frac{(m-n-1)(m-n-2)}{\underline{|2}}\cdot\frac{1}{(n+2)(x-1)^{n+2}}+\&c$

9. $\frac{1}{nx^r}+\frac{r}{n(n+1)}\cdot\frac{1}{x^{r+1}}+\frac{r(r+1)}{n(n+1)(n+2)}\cdot\frac{1}{x^{r+2}}+\&c$

$= \frac{1}{n(x-1)^r}-\frac{r}{\underline{|1}}\cdot\frac{1}{(n+1)(x-1)^{r+1}}+\frac{r(r+1)}{\underline{|2}}\cdot\frac{1}{(n+2)(x-1)^{r+2}}+\&c$

10. $\frac{1}{nx}+\frac{\underline{|1}}{n(n+1)x^2}+\frac{\underline{|2}}{n(n+1)(n+2)x^3}+\&c$

$= \frac{1}{n(x-1)}-\frac{1}{(n+1)(x-1)}+\frac{1}{(n+2)(x-1)^3}-\&c$

11. $(1-x)^{a+b}\left\{1+\frac{ab}{1.n}x+\frac{a(a+1)b(b+1)}{1.2\,n(n+1)}x^2+\&c\right\}$

$= (1-x)^n\left\{1+\frac{(n-a)(n-b)}{1.n}x+\frac{(n-a)(n-a+1)(n-b)(n-b+1)}{1.2\,n(n+1)}x^2+\&c\right.$

Sol. Apply XIII 4 twice.

12. $\frac{\underline{|x+y+n}}{\underline{|x+n}\,\underline{|y+n}}+\frac{PQ}{\underline{|1}}\cdot\frac{\underline{|x+y+n+1}}{\underline{|x+n+1}\,\underline{|y+n+1}}+\frac{P(P-1)Q(Q-1)}{\underline{|2}}\frac{\underline{|x+y+n+2}}{\underline{|x+n+2}\,\underline{|y+n+2}}+\&c$

$= \frac{\underline{|P+Q+n}}{\underline{|P+n}\,\underline{|Q+n}}+\frac{xy}{\underline{|1}}\frac{\underline{|P+Q+n+1}}{\underline{|P+n+1}\,\underline{|Q+n+1}}+\frac{x(x-1)y(y-1)}{\underline{|2}}\frac{\underline{|P+Q+n+2}}{\underline{|P+n+2}\,\underline{|Q+n+2}}+\&c$

Sol. By XIII we have,

$\frac{\underline{|x+y+n}}{\underline{|x+n}\,\underline{|y+n}}=\frac{1}{\underline{|n}}+\frac{xy}{\underline{|1}}\frac{1}{\underline{|n+1}}+\frac{x(x-1)y(y-1)}{\underline{|2}}\frac{1}{\underline{|n+2}}$

$\frac{PQ}{\underline{|1}}\cdot\frac{\underline{|x+y+n+1}}{\underline{|x+n+1}\,\underline{|y+n+1}}=\frac{PQ}{\underline{|1}}\cdot\frac{1}{\underline{|n+1}}+\frac{PQ}{\underline{|1}}\cdot\frac{xy}{\underline{|1}}\frac{1}{\underline{|n+2}}+\frac{PQ}{\underline{|1}}$ [illegible]

$$1 - \frac{n}{\lfloor 1} \cdot \frac{1}{2} \cdot \frac{1}{3} + \frac{n(n-1)}{\lfloor 2} \cdot \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5} - \&c$$

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{\lfloor n}{\lfloor n+\frac{1}{2}} \left\{ 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 + \&\text{ to } n+1 \text{ terms} \right\}$$

$$\frac{\pi}{2} \left\{ \frac{1}{n} + \left(\frac{1}{2}\right)^2 \frac{1}{n+1} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{1}{n+2} + \&c \right\}$$

$$= \frac{1}{n} + \frac{n+\frac{1}{2}}{n(n+1)} \cdot \frac{1}{3} + \frac{(n+\frac{1}{2})(n+1\frac{1}{2})}{n(n+1)(n+2)} \cdot \frac{1}{5} + \&c$$

$$\frac{1}{n} + \frac{x}{\lfloor 1} \cdot \frac{y}{z} \cdot \frac{1}{n+1} + \frac{x(x-1)}{\lfloor 2} \cdot \frac{y(y-1)}{z(z+1)} \cdot \frac{1}{n+2} + \&c$$

$$= \frac{\lfloor x \, \lfloor n-1}{\lfloor x+n} \left\{ 1 + \frac{n}{\lfloor 1} \cdot \frac{y+z}{z} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{(y+z)(y+z+1)}{z(z+1)} + \&c \text{ to } x+1 \text{ terms} \right\}$$

$$\frac{P(P-1)}{\lfloor\underline{2}}\frac{Q(Q-1)}{}\;\frac{\lfloor\underline{x+y+n+2}}{\lfloor\underline{x+n+2}\,\lfloor\underline{y+n+2}} = \frac{P(P-1)}{\lfloor\underline{2}}\frac{Q(Q-1)}{}\cdot\frac{1}{\lfloor\underline{n+2}} + \&c$$

&c &c &c &c &c

Add up all the results.

13. $$\frac{1}{p+n} + \frac{x}{1}\cdot\frac{y}{n}\cdot\frac{1}{p+n+1} + \frac{x(x-1)}{\lfloor\underline{2}}\cdot\frac{y(y-1)}{n(n+1)}\cdot\frac{1}{p+n+2} + \&c$$

$$= \frac{\lfloor\underline{n-1}\,\lfloor\underline{x+y+n}}{\lfloor\underline{x+n}\,\lfloor\underline{y+n}} - p\cdot\frac{\lfloor\underline{n-1}\,\lfloor\underline{x+y+n+1}}{\lfloor\underline{x+n+1}\,\lfloor\underline{y+n+1}} + p(p-1)\frac{\lfloor\underline{n-1}\,\lfloor\underline{x+y+n+2}}{\lfloor\underline{x+n+2}\,\lfloor\underline{y+n+2}} - \&c$$

14. $$\frac{\pi}{4}\left\{\frac{1}{n+1} + \left(\frac{1}{2}\right)^2\frac{1}{n+2} + \left(\frac{1.3}{2.4}\right)^2\frac{1}{n+3} + \left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{n+4} + \&c\right\}$$

$$= 1 - \frac{n}{1}\cdot\left(\frac{2}{3}\right)^2 + \frac{n(n-1)}{\lfloor\underline{2}}\cdot\left(\frac{2.4}{3.5}\right)^2 - \frac{n(n-1)(n-2)}{\lfloor\underline{3}}\left(\frac{2.4.6}{3.5.7}\right)^2 + \&c$$

$$= \left(\frac{2.4.6.\ldots\ldots.2n}{3.5.7.\ldots\ldots.2n+1}\right)^2\left\{1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 + \&c \text{ to } n+1 \text{ terms}\right\}$$

Cor. 1. $$\frac{\pi}{4}\left\{1 + \left(\frac{1}{2}\right)^2\frac{1}{3} + \left(\frac{1.3}{2.4}\right)^2\frac{1}{5} + \left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{7} + \&c\right\}$$

$$= \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c$$

2. $$\pi n\left\{\frac{1}{n} + \left(\frac{1}{2}\right)^2\frac{1}{n+1} + \left(\frac{1.3}{2.4}\right)^2\frac{1}{n+2} + \&c\right\} - \log_e n$$

is finite when $n = \infty$

15. $$\frac{1}{y+n} - \frac{x}{n}\cdot\frac{1}{y+n+1} + \frac{x(x-1)}{n(n+1)}\cdot\frac{1}{y+n+2} - \&c$$

$$= \frac{1}{x+n} - \frac{y}{n}\cdot\frac{1}{x+n+1} + \frac{y(y-1)}{n(n+1)}\cdot\frac{1}{x+n+2} - \&c$$

16. $$\frac{1}{\{\phi(x)\}^n}\left[1 + \frac{n}{1}\cdot\frac{m}{2m}\left\{\frac{\phi(x)-\phi(-x)}{\phi(x)}\right\} + \frac{n(n+1)}{\lfloor\underline{2}}\cdot\frac{m(m+1)}{2m(2m+1)}\left\{\frac{\phi(x)-\phi(-x)}{\phi(x)}\right\}^2 + \&c\right]$$

is always an even function of x.

17. $$1 + \frac{n}{1}\cdot\frac{m}{2m}\cdot\left(\frac{2x}{1+x}\right) + \frac{n(n+1)}{\lfloor\underline{2}}\cdot\frac{m(m+1)}{2m(2m+1)}\left(\frac{2x}{1+x}\right)^2 + \&c$$

$$= (1+x)^n\left\{1 + \frac{n(n+1)}{2(2m+1)}x^2 + \frac{n(n+1)(n+2)(n+3)}{2.4.(2m+1)(2m+3)}x^4 + \&c\right\}$$

18. $$1 + \frac{n}{1}\cdot\frac{m}{2m}\cdot\frac{4x}{(1+x)^2} + \frac{n(n+1)}{\lfloor\underline{2}}\cdot\frac{m(m+1)}{2m(2m+1)}\left\{\frac{4x}{(1+x)^2}\right\}^2 + \&c$$

$$\phi(0) + \frac{2\phi'(0)}{\lfloor 1}\cdot\frac{m}{2m} + \frac{2^2\phi''(0)}{\lfloor 2}\cdot\frac{m(m+1)}{2m(2m+1)} + \&c$$

$$= \phi(1) + \frac{\phi''(1)}{2(2m+1)} + \frac{\phi^{iv}(1)}{2.4.(2m+1)(2m+3)} + \&c$$

If $m(m-1) = 2p$

$$e^{-mx}\left\{1 + \frac{1}{2}\cdot\frac{m}{\lfloor 1}(1-e^{-2x}) + \frac{1.3}{2.4}\cdot\frac{m(m+1)}{\lfloor 2}(1-e^{-2x})^2 + \&c\right\}$$

$$= 1 + \frac{A_1 x^2}{2(\lfloor 1)^2} + \frac{A_2 x^4}{2^2(\lfloor 2)^2} + \frac{A_3 x^6}{2^3(\lfloor 3)^2} + \&c$$

where $A_n = p^n - \frac{n(n-1)}{\lfloor 3}p^{n-1} + \frac{n(n-1)(n-2)(3n-1)}{\lfloor 5}p^{n-2}$

$- \ldots \&c + (-1)^{n-1} 2p\cdot\frac{\lfloor n-1\,(2^{2n}-1)B_{2n}}{1.3.5\ldots(2n-1)} = \phi(p)$

$$\phi(1) = \frac{\lfloor n}{1.3.5\ldots(2n-1)}$$

$$\left\{1 + \left(\tfrac{1}{2}\right)^2 x + \ldots\right\}^2 = \cfrac{1}{1 - \cfrac{x}{2 - \cfrac{3x}{8 - \cfrac{5x}{2 - \cfrac{17x}{40 - \cfrac{23x}{2 - \cfrac{1395x}{3128 - \&c}}}}}}}$$

$$=(1+x)^{2n}\left\{1+\frac{n}{\lfloor 1}\cdot\frac{n-m+\frac{1}{2}}{m+\frac{1}{2}}x^2+\frac{n(n+1)}{\lfloor 2}\cdot\frac{(n-m+\frac{1}{2})(n-m+1\frac{1}{2})}{(m+\frac{1}{2})(m+1\frac{1}{2})}x^4+\&c.\right\}$$

19. $$1+\frac{2x}{\lfloor 1}\cdot\frac{m}{2m}+\frac{(2x)^2}{\lfloor 2}\cdot\frac{m(m+1)}{2m(2m+1)}+\frac{(2x)^3}{\lfloor 3}\times\frac{m(m+1)(m+2)}{2m(2m+1)(2m+2)}+\&c$$

$$=e^x\left\{1+\frac{x^2}{2}\cdot\frac{1}{2m+1}+\frac{x^4}{2\cdot 4}\cdot\frac{1}{(2m+1)(2m+3)}+\&c\right\}$$

Cor. $$1+\frac{1}{2}\cdot\frac{x}{\lfloor 1}+\frac{1.3}{2.4}\cdot\frac{x^2}{\lfloor 2}+\frac{1.3.5}{2.4.6}\cdot\frac{x^3}{\lfloor 3}+\&c$$

$$=e^{\frac{x}{2}}\left\{1+\frac{x^2}{4^2}+\frac{x^4}{4^2\cdot 8^2}+\frac{x^6}{4^2\cdot 8^2\cdot 12^2}+\&c\right\}$$

Ex. 1. $$1-\frac{1.3}{2.4}\cdot\frac{\pi^2}{\lfloor 2}+\frac{1.3.5.7}{2.4.6.8}\cdot\frac{\pi^4}{\lfloor 4}-\&c=0.$$

2. $$1-\frac{1.3}{1^2.2^2}\pi^2+\frac{1.3.5.7}{1^2.2^2.3^2.4^2}\pi^4-\&c$$

$$=-\left(1-\frac{\pi^2}{2^2}+\frac{\pi^4}{2^2.4^2}-\frac{\pi^6}{2^2.4^2.6^2}+\&c\right)$$

3. $$1-\frac{1.3}{3.6}\cdot\frac{\pi^2}{\lfloor 2}+\frac{1.3.5.7}{3.6.9.12}\cdot\frac{\pi^4}{\lfloor 4}-\&c$$

$$=\frac{1}{2}\left(1-\frac{\pi^2}{6^2}+\frac{\pi^4}{6^2.12^2}-\frac{\pi^6}{6^2.12^2.18^2}+\&c\right)$$

20. i. $$1+\left(\frac{1}{2}\right)^2\frac{4x}{(1+x)^2}+\left(\frac{1.3}{2.4}\right)^2\left\{\frac{4x}{(1+x)^2}\right\}^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left\{\frac{4x}{(1+x)^2}\right\}^3$$

$$=(1+x)\left\{1+\left(\frac{1}{2}\right)^2x^2+\left(\frac{1.3}{2.4}\right)^2x^4+\left(\frac{1.3.5}{2.4.6}\right)^2x^6+\&c\right\}$$

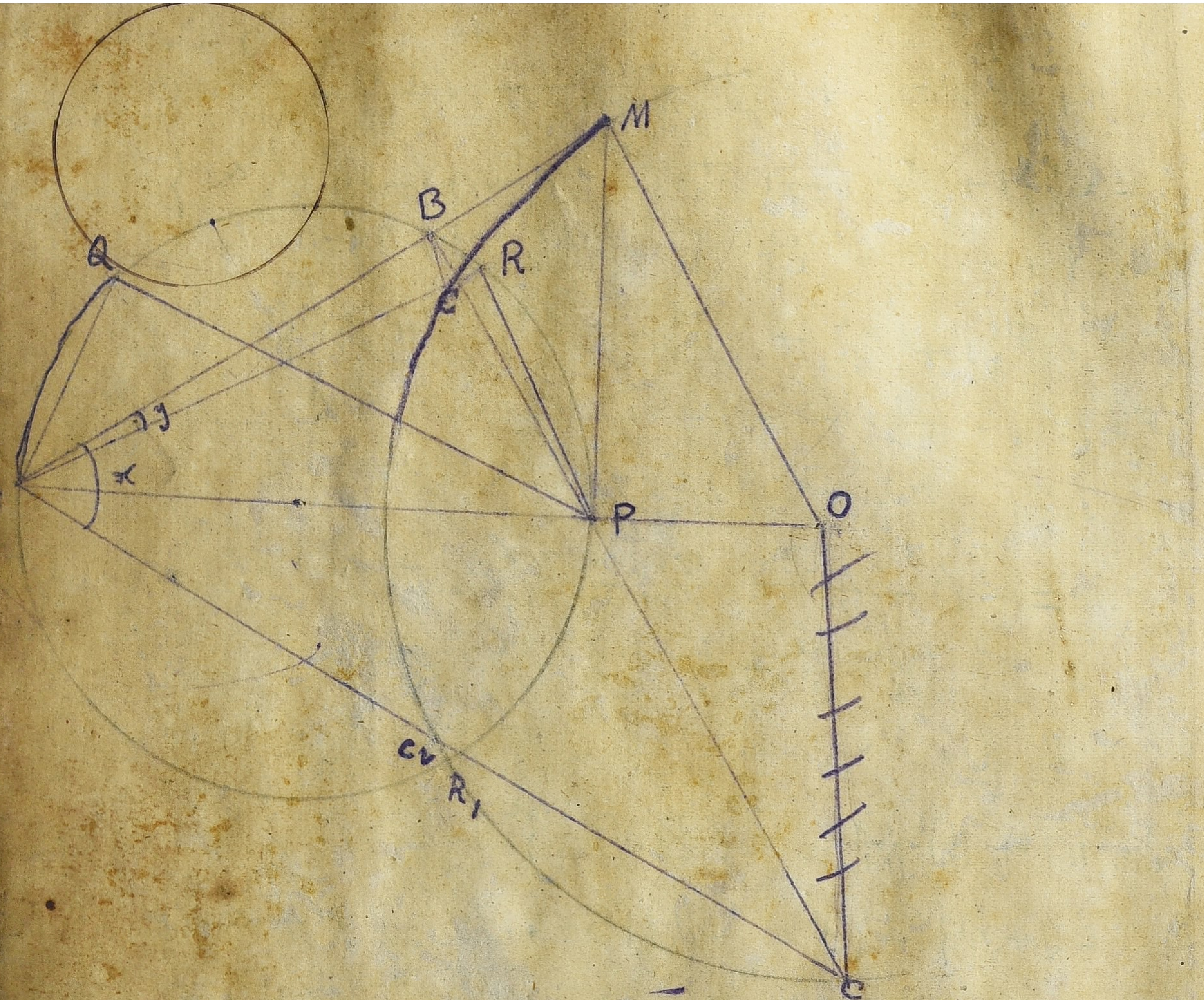
ii. $$1+\left(\frac{1}{2}\right)^2\frac{2x}{1+x}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{2x}{1+x}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left(\frac{2x}{1+x}\right)^3+\&c$$

$$=\sqrt{1+x}\left(1+\frac{1.3}{4^2}x^2+\frac{1.3.5.7}{4^2.8^2}x^4+\frac{1.3.5.7.9.11}{4^2.8^2.12^2}x^6+\&c\right\}$$

Ex. 1. $$1+\frac{n}{\lfloor 1}\cdot\frac{m}{2m}\cdot\frac{4x}{(1+x)^2}+\frac{n(n+1)}{\lfloor 2}\cdot\frac{m(m+1)}{2m(2m+1)}\left\{\frac{4x}{(1+x)^2}\right\}^2+\&c$$

$$=\frac{(1-x^2)^{2m}}{(1-x)^{2n}}\left\{1+\frac{2m-n}{\lfloor 1}\cdot\frac{m-n+\frac{1}{2}}{m+\frac{1}{2}}x^2+\frac{(2m-n)(2m-n+1)}{\lfloor 2}\cdot\frac{(m-n+\frac{1}{2})(m-n+1\frac{1}{2})}{(m+\frac{1}{2})(m+1\frac{1}{2})}x^4+\&c\right\}$$

Sol. apply XIII ii in R. H. S of XIII 18.



$$\frac{\frac{\beta}{\gamma}x + \frac{\alpha}{\lfloor 1}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha-1)}{\lfloor 2}\cdot\frac{\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}x^3 + \&c}{1 + \frac{\alpha}{\lfloor 1}\cdot\frac{\beta}{\gamma}x + \frac{\alpha(\alpha+1)}{\lfloor 2}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \&c}$$

$$= \cfrac{\beta x}{\gamma - (\alpha+\beta+1)x + \cfrac{(\beta+1)(\alpha+\gamma+1)x}{\gamma+1-(\alpha+\beta+2)x + \cfrac{(\beta+2)(\alpha+\gamma+2)x}{\gamma+2-(\alpha+\beta+3)x + \&c}}}$$

2. $1+\frac{n}{\underline{|1}}\cdot\frac{m}{2m}\cdot\frac{2x}{1+x}+\frac{n(n+1)}{\underline{|2}}\cdot\frac{m(m+1)}{2m(2m+1)}\left(\frac{2x}{1+x}\right)^2+\&c$

$$=\frac{(1-x^2)^m}{(1-x)^n}\left\{1+\frac{(2m-n)(2m-n+1)}{2.(2m+1)}x^2+\frac{(2m-n)(2m-n+1)}{2\cdot4}\times\frac{(2m-n+2)(2m-n+3)}{(2m+1)(2m+3)}x^4+\&c\right\}$$

Sol. Apply $\underline{\text{XIII}}$ 11 in R.H.S of $\underline{\text{XIII}}$ 17

3. $1+\frac{n(n+1)}{2(2m+1)}\cdot\frac{4x}{(1+x)^2}+\frac{n(n+1)(n+2)(n+3)}{2\cdot4(2m+1)(2m+3)}\left\{\frac{4x}{(1+x)^2}\right\}^2+\&c$

$$=(1+x)^n\left\{1+\frac{n}{\underline{|1}}\cdot\frac{n-m+\frac{1}{2}}{m+\frac{1}{2}}x+\frac{n(n+1)}{\underline{|2}}\cdot\frac{(n-m+\frac{1}{2})(n-m+1\frac{1}{2})}{(m+\frac{1}{2})(m+1\frac{1}{2})}x^2+\&c\right.$$

Sol. Combine the results of $\underline{\text{XIII}}$ 17 & 18.

4. $1+\frac{n(n+1)}{1^2}\cdot\frac{x}{(1+x)^2}+\frac{n(n+1)(n+2)(n+3)}{1^2\cdot2^2}\cdot\frac{x^2}{(1+x)^4}+\&c$

$$=(1+x)^n\left\{1+\frac{n^2}{(\underline{|1})^2}x+\frac{n^2(n+1)^2}{(\underline{|2})^2}x^2+\frac{n^2(n+1)^2(n+2)^2}{(\underline{|3})^2}x^3+\&c\right\}$$

21. $\left\{1+\frac{x}{\underline{|1}}\cdot\frac{1}{m}+\frac{x^2}{\underline{|2}}\cdot\frac{1}{m(m+1)}+\frac{x^3}{\underline{|3}}\cdot\frac{1}{m(m+1)(m+2)}+\&c\right\}$

$\times\left\{1+\frac{x}{\underline{|1}}\cdot\frac{1}{n}+\frac{x^2}{\underline{|2}}\cdot\frac{1}{n(n+1)}+\frac{x^3}{\underline{|3}}\cdot\frac{1}{n(n+1)(n+2)}+\&c\right\}$

$$=1+\frac{x}{\underline{|1}}\cdot\frac{m+n}{mn}+\frac{x^2}{\underline{|2}}\cdot\frac{(m+n+1)(m+n+2)}{m(m+1)\,n(n+1)}+\frac{x^3}{\underline{|3}}\cdot\frac{(m+n+2)(m+n+3)(m+n+4)}{m(m+1)(m+2)\,n(n+1)(n+2)}+\&c$$

Sol. From   we have,

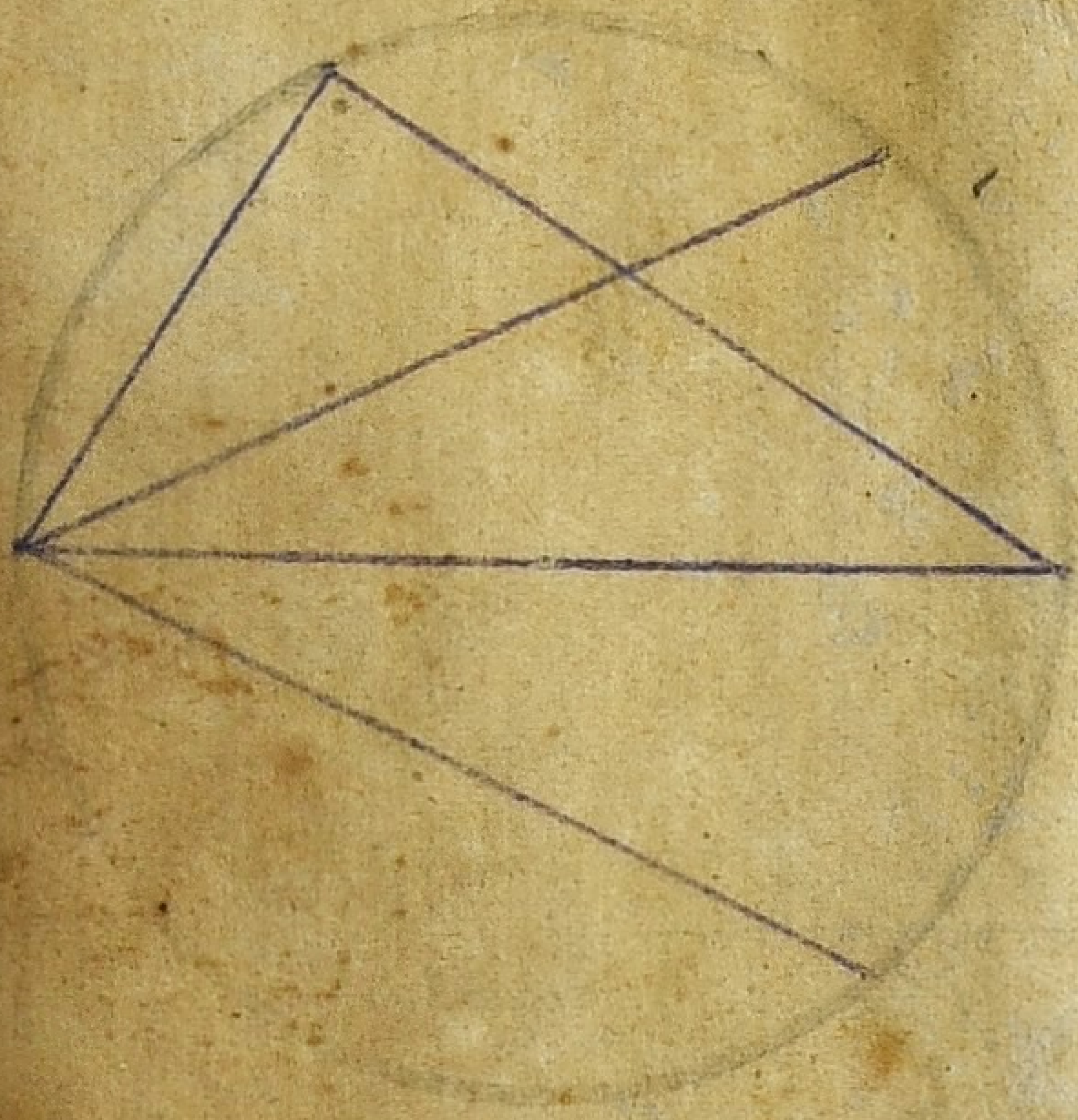
$$\frac{\underline{|m}\,x^t}{\underline{|m+t}\,\underline{|t}}+\frac{1}{\underline{|1}(n+1)}\cdot\frac{x^t\,\underline{|m}}{\underline{|m+t-1}\,\underline{|t-1}}+\frac{1}{\underline{|2}}\cdot\frac{x^t}{(n+1)(n+2)}\cdot\frac{\underline{|m}}{\underline{|m+t-2}\,\underline{|t-2}}$$

$$=\frac{\underline{|n}\,\underline{|m+2t+n}\,\underline{|m}\,x^t}{\underline{|m+t+n}\,\underline{|t+n}\,\underline{|m+t}\,\underline{|t}}$$

Equate the series which [illegible] are their [illegible]

In a similar manner $\underline{\text{XIII}}$ 22, [illegible]

$$\cfrac{1}{1+\cfrac{a_1x}{1+\cfrac{a_2x}{1+\cfrac{a_3x}{1+\&c}}}} = 1 - A_1x + A_2x^2 - A_3x^3 + \&c$$



Let $P_n = a_1a_2\cdots a_{n-1}(a_1+a_2+\cdots a_n)$

$P_1 = A_1$

$P_2 = A_2$

$P_3 = A_3 - a_1A_2$

$P_4 = A_4 - (a_1+a_2)A_3$

$P_5 = A_5 - (a_1+a_2+a_3)A_4 + a_1a_3A_3$

$P_6 = A_6 - (a_1+a_2+a_3+a_4)A_5 + (a_1a_3+a_1a_4+a_2a_4)A_4$

$P_n = \phi_0(n)A_n - \phi_1(n)A_{n-1} + \phi_2(n)A_{n-2} - \&c$

where $\phi_r(n+1) - \phi_r(n) = a_{n-1}\phi_{r-1}(n-1)$.

$$\cfrac{1}{1+b_1x+\cfrac{a_1x}{1+b_2x+\cfrac{a_2x}{1+b_3x+\&c}}} = 1 - A_1x + A_2x^2 - A_3x^3 + \&c$$

$P_n = a_1a_2a_3\cdots a_{n-1}(\overline{a_1+b_1} + \overline{a_2+b_2} + \cdots + \overline{a_n+b_n})$

$= \phi_0(n)A_n - \phi_1(n)A_{n-1} + \phi_2(n)A_{n-2} - \&c$

where $\phi_r(n+1) - \phi_r(n) = b_n\phi_{r-1}(n) + a_{n-1}\phi_{r-1}(n-1)$.

$D_{n-1} = \phi_0(n) + x\phi_1(n) + x^2\phi_2(n) + x^3\phi_3(n) + \&c$

Cor. $e^x\left\{1+\frac{x^2}{2}\cdot\frac{1}{2n+1}+\frac{x^4}{2\cdot4}\cdot\frac{1}{(2n+1)(2n+3)}+\&c\right\}$

$=1+\frac{2x}{\underline{|1}}\cdot\frac{n}{2n}+\frac{(2x)^2}{\underline{|2}}\cdot\frac{n(n+1)}{2n(2n+1)}+\&c$

2. $\left\{1+\frac{x}{4}\cdot\frac{1}{2n+1}+\frac{x^2}{4\cdot8}\cdot\frac{1}{(2n+1)(2n+3)}+\&c\right\}^2$

$=1+\frac{x}{\underline{|1}}\cdot\frac{n}{2n}\cdot\frac{1}{2n+1}+\frac{x^2}{\underline{|2}}\cdot\frac{n(n+1)}{2n(2n+1)}\cdot\frac{1}{(2n+1)(2n+3)}+\&c$

Ex. 1. $1+\frac{1}{2}\cdot\frac{x}{\underline{|1}}+\frac{1.3}{2.4}\cdot\frac{x^2}{\underline{|2}}+\frac{1.3.5}{2.4.6}\cdot\frac{x^3}{\underline{|3}}+\&c$

$=e^{\frac{x}{2}}\left(1+\frac{x^2}{4^2}+\frac{x^4}{4^2 8^2}+\frac{x^6}{4^2 8^2 12^2}+\&c\right)$

2. $1+\frac{1}{2}\cdot\frac{x}{(\underline{|1})^2}+\frac{1.3}{2.4}\cdot\frac{x^2}{(\underline{|2})^2}+\frac{1.3.5}{2.4.6}\cdot\frac{x^3}{(\underline{|3})^2}+\&c$

$=\left(1+\frac{x}{2^2}+\frac{x^2}{2^2 4^2}+\frac{x^4}{2^2 4^2 6^2}+\frac{x^6}{2^2 4^2 6^2 8^2}+\&c\right)^2$

22. $\left\{1+\frac{x}{\underline{|1}}\cdot\frac{1}{m+1}\cdot\frac{1}{n+1}+\frac{x^2}{\underline{|2}}\cdot\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n+1)(n+2)}+\&c\right\}$

$\times\left\{1-\frac{x}{\underline{|1}}\cdot\frac{1}{m+1}\cdot\frac{1}{n+1}+\frac{x^2}{\underline{|2}}\cdot\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n+1)(n+2)}-\&c\right\}$

$=1-\frac{x^2}{\underline{|1}}\cdot\frac{m+n+3}{(m+1)(n+1)}\cdot\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n+1)(n+2)}$

$+\frac{x^4}{\underline{|2}}\cdot\frac{(m+n+5)(m+n+6)}{(m+1)(m+2)(n+1)(n+2)}\cdot\frac{1}{(m+1)(m+2)(m+3)(m+4)}$

$\times\frac{1}{(n+1)(n+2)(n+3)(n+4)}-\frac{x^6}{\underline{|3}}\cdot\frac{(m+n+7)(m+n+8)(m+n+9)}{(m+1)(m+2)(m+3)(n+1)(n+2)(n+3)}$

$\times\frac{1}{(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)}\times\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}$

$+\&c$

23. $\left\{1+\frac{x}{\underline{|1}}\cdot\frac{1}{(m+n+1)}\cdot\frac{1}{n+1}+\frac{x^2}{\underline{|2}}\cdot\frac{1}{(m+n+1)(m+n+2)}\cdot\frac{1}{(n+1)(n+2)}+\&c\right\}$

$\left\{1+\frac{x}{\underline{|1}}\cdot\frac{1}{m+1}\cdot\frac{1}{n-1}+\frac{x^2}{\underline{|2}}\cdot\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n-1)(n-2)}+\&c\right\}$

$=1+\frac{x}{\underline{|1}}\cdot\frac{2m+n+3}{m+n+1}\cdot\frac{1}{m+1}\cdot\frac{n}{n^2-1^2}+\frac{x^2}{\underline{|2}}\cdot\frac{\text{[illegible]}}{(m+n+1)\text{[illegible]}}$

$\times\frac{1}{(m+1)(m+2)}\cdot\frac{1}{n^2-2^2}+$

$$n\left\{1+\frac{x^n}{\lfloor n}+\frac{x^{2n}}{\lfloor 2n}+\frac{x^{3n}}{\lfloor 3n}+ \&c\right\}$$

$$= e^x + e^{x\cos\frac{2\pi}{n}}\cos\left(x\sin\frac{2\pi}{n}\right) + e^{x\cos\frac{4\pi}{n}}\cos\left(x\sin\frac{4\pi}{n}\right)$$

$$+ \&c \text{ to } n \text{ terms.}$$

$$\frac{x^h}{\lfloor h} + \left(\frac{x^{h+n}}{\lfloor h+n} + \frac{x^{h-n}}{\lfloor h-n}\right) + \left(\frac{x^{h+2n}}{\lfloor h+2n} + \frac{x^{h-2n}}{\lfloor h-2n}\right) + \&c$$

$$= 1 + \left(\frac{x^n}{\lfloor n} + \frac{x^{-n}}{\lfloor -n}\right) + \left(\frac{x^{2n}}{\lfloor 2n} + \frac{x^{-2n}}{\lfloor -2n}\right) + \&c$$

$$\left\{6m^2 + (3m^3 - m)\right\}^3 + \left\{6m^2 - (3m^3 - m)\right\}^3$$

$$= \left\{6m^2(3m^2+1)\right\}^2.$$

$$\left\{m^7 - 3m^4(1+p) + m\left(3\cdot\overline{1+p}^2 - 1\right)\right\}^3$$

$$+ \left\{2m^6 - 3m^3(1+2p) + (1+3p+3p^2)\right\}^3$$

$$+ \left\{m^6 - (1+3p+3p^2)\right\}^3$$

$$= \left\{m^7 - 3m^4 p + m(3p^2 - 1)\right\}^3$$

$$\int_{-\infty}^{\infty}\frac{\phi(x)}{\lfloor x}\,dx = \phi(1) + \frac{\phi(1)}{\lfloor 1} + \frac{\phi(2)}{\lfloor 2} + \frac{\phi(3)}{\lfloor 3} + \&c$$

$$\int_{-\infty}^{\infty}\frac{a^x}{\lfloor x}\,dx = e^a \qquad \int_{-\infty}^{\infty}\frac{\lfloor n}{\lfloor x\,\lfloor n-x}\,a^x\,dx = (1+a)^n.$$

$$+ \frac{x^3}{\underline{|3}} \frac{(2m+n+5)(2m+n+7)(2m+n+9)}{(m+n+1)(m+n+2)(m+n+3)} \frac{1}{(m+1)(m+2)(m+3)} \times$$

$$\frac{n}{(n^2-1^2)(n^2-3^2)} + \frac{x^4}{\underline{|4}} \frac{(2m+n+6)(2m+n+8)(2m+n+10)(2m+n+12)}{(m+n+1)(m+n+2)(m+n+3)(m+n+4)}$$

$$\times \frac{1}{(m+1)(m+2)(m+3)(m+4)} \frac{1}{(n^2-2^2)(n^2-4^2)} + \&c$$

24. $$\left\{1 + \frac{x}{\underline{|1}} \cdot \frac{m}{n+1} + \frac{x^2}{\underline{|2}} \cdot \frac{m(m-1)}{(n+1)(n+2)} + \&c\right\}$$

$$\times \left\{1 - \frac{x}{\underline{|1}} \cdot \frac{m}{n+1} + \frac{x^2}{\underline{|2}} \cdot \frac{m(m-1)}{(n+1)(n+2)} - \&c\right\}$$

$$= 1 - \frac{x^2}{\underline{|1}} \frac{m+n+1}{(n+1)(n+2)} \cdot \frac{m}{n+1} + \frac{x^4}{\underline{|2}} \cdot \frac{(m+n+1)(m+n+2)}{(n+1)(n+2)(n+3)(n+4)}$$

$$\times \frac{m(m-1)}{(n+1)(n+2)} - \&c$$

25. $$\left\{1 + \frac{x}{\underline{|1}} mn + \frac{x^2}{\underline{|2}} m(m-1)n(n-1) + \&c\right\}$$

$$\times \left\{1 - \frac{x}{\underline{|1}} mn + \frac{x^2}{\underline{|2}} m(m-1)n(n-1) - \&c\right\}$$

$$= 1 - \frac{x^2}{\underline{|1}} mn(m+n-1) + \frac{x^4}{\underline{|2}} m(m-1)n(n-1)(m+n-2)(m+n$$

$$- \frac{x^6}{\underline{|3}} m(m-1)(m-2)n(n-1)(n-2)(m+n-3)(m+n-4)(m+n-5) + \&$$

26. $$\left\{1 + \frac{x}{\underline{|1}} \cdot \frac{m}{n+1} + \frac{x^2}{\underline{|2}} \frac{m(m-1)}{(n+1)(n+2)} + \&c\right\}$$

$$\times \left\{1 + \frac{x}{\underline{|1}} \cdot \frac{m+n}{n-1} + \frac{x^2}{\underline{|2}} \cdot \frac{(m+n)(m+n-1)}{(n-1)(n-2)} + \&c\right\}$$

$$= 1 + \frac{x}{\underline{|1}} (2m+n+1) \frac{n}{n^2-1^2} + \frac{x^2}{\underline{|2}} (2m+n)(2m+n+2) \frac{1}{n^2-2^2}$$

$$+ \frac{x^3}{\underline{|3}} (2m+n-1)(2m+n+1)(2m+n+3) \frac{n}{(n^2-1^2)(n^2-3^2)} + \&c$$

Ex. 1. $$\left(1 + \frac{x^3}{\underline{|3}} + \frac{x^6}{\underline{|6}} + \frac{x^9}{\underline{|9}} + \&c\right)\left(1 - \frac{x^3}{\underline{|3}} + \frac{x^6}{\underline{|6}} - \frac{x^9}{\underline{|9}} + \&c\right)$$

$$= \frac{1}{3} + \frac{2}{3}\left\{1 - \frac{(3x^2)^3}{\underline{|6}} + \frac{(3x^2)^6}{\underline{|12}} - \frac{(3x^2)^9}{\underline{|18}} + \&c\right\}$$

2. $$\left\{1 + \frac{x}{(\underline{|1})^3} + \frac{x^2}{(\underline{|2})^3} + \frac{x^3}{(\underline{|3})^3} + \&c\right\}\left\{1 - \frac{x}{(\underline{|1})^3} + \frac{x^2}{(\underline{|2})^3} - \&c\right\}$$

$$= 1 - \frac{\underline{|3}}{(\underline{|1}\,\underline{|2})^3} x^2 + \frac{\underline{|6}}{(\underline{|2}\,\underline{|4})^3} x^4 - \frac{\underline{|9}}{(\underline{|3}\,\underline{|6})^3} x^6 + \&c$$

$$1 + \frac{1+x}{\lfloor 1}\cdot\frac{mn}{m+n+1} + \frac{(1+x)^2}{\lfloor 2}\cdot\frac{m(m+1)\,n(n+1)}{(m+n+1)(m+n+2)} + \&c$$

$$= \sqrt{\pi}\,\frac{\left\lfloor \frac{m+n-1}{2}\right.}{\left\lfloor \frac{m-1}{2}\right.\left\lfloor \frac{n-1}{2}\right.}\left\{1 + \frac{x^2}{\lfloor 2}mn + \frac{x^4}{\lfloor 4}m(m+2) \times n(n+2) + \&c\right\}$$

$$+ 2\sqrt{\pi}\,\frac{\left\lfloor \frac{m+n-1}{2}\right.}{\left\lfloor \frac{m}{2}-1\right.\left\lfloor \frac{n}{2}-1\right.}\left\{\frac{x}{\lfloor 1} + \frac{x^3}{\lfloor 3}(m+1)(n+1) + \frac{x^5}{\lfloor 5} \times (m+1)(m+3)(n+1)(n+3) + \&c\right\}$$

I = α, II = β, IV = γ then

$$\frac{\sqrt{3}\,\sqrt[6]{\beta(1-\beta)}}{\sqrt[3]{\alpha(1-\gamma)} - \sqrt[3]{\gamma(1-\alpha)}} = \frac{\left\{1 + \frac{1.2}{3^2}\alpha + \&c\right\}\left\{1 + \frac{1.2}{3^2}\gamma + \&c\right\}}{\left\{1 + \frac{1.2}{3^2}\beta + \&c\right\}^2}.$$

3. $\left(x + \frac{x^4}{\underline{|4}} + \frac{x^7}{\underline{|7}} + \frac{x^{10}}{\underline{|10}} + \&c\right)\left(x - \frac{x^4}{\underline{|4}} + \frac{x^7}{\underline{|7}} - \frac{x^{10}}{\underline{|10}} + \&c\right)$

$= \frac{2}{3}\left\{\frac{(3x^2)}{\underline{|2}} - \frac{(3x^2)^4}{\underline{|8}} + \frac{(3x^2)^7}{\underline{|14}} - \frac{(3x^2)^{10}}{\underline{|20}} + \&c\right\}$

4. $\cos x \cdot \cosh x = 1 - \frac{(2x^2)^2}{\underline{|4}} + \frac{(2x^2)^4}{\underline{|8}} - \frac{(2x^2)^6}{\underline{|12}} + \&c$

5. $\sin x \sinh x = \frac{(2x^2)}{\underline{|2}} - \frac{(2x^2)^3}{\underline{|6}} + \frac{(2x^2)^5}{\underline{|10}} - \&c$

6. $\left\{1 + \frac{x}{(\underline{|1})^2} + \frac{x^2}{(\underline{|2})^2} + \frac{x^3}{(\underline{|3})^2} + \&c\right\}\left\{1 - \frac{x}{(\underline{|1})^2} + \frac{x^2}{(\underline{|2})^2} - \frac{x^3}{(\underline{|3})^2} + \&c\right\}$

$= 1 - \frac{x^2}{\underline{|2}} \cdot \frac{1}{(\underline{|1})^2} + \frac{x^4}{\underline{|4}} \cdot \frac{1}{(\underline{|2})^2} - \frac{x^6}{\underline{|6}} \cdot \frac{1}{(\underline{|3})^2} + \&c$

7. $\left\{1 + \frac{1}{2} \cdot \frac{x}{\underline{|1}} + \frac{1.3}{2.4} \cdot \frac{x^2}{\underline{|2}} + \frac{1.3.5}{2.4.6} \frac{x^3}{\underline{|3}} + \&c\right\}\left\{1 - \frac{1}{2} \cdot \frac{x}{\underline{|1}} + \frac{1.3}{2.4} \cdot \frac{x^2}{\underline{|2}} \&c\right\}$

$= 1 + \frac{1}{2} \cdot \frac{x^2}{2^2} + \frac{1.3}{2.4} \cdot \frac{x^4}{2^2 4^2} + \frac{1.3.5}{2.4.6} \cdot \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \&c$

8. $\left\{\frac{1}{n} + \frac{x}{n(n+1)} + \frac{x^2}{n(n+1)(n+2)} + \frac{x^3}{n(n+1)(n+2)(n+3)} + \&c\right\}$

$\times \left\{\frac{1}{n} - \frac{x}{n(n+1)} + \frac{x^2}{n(n+1)(n+2)} - \frac{x^3}{n(n+1)(n+2)(n+3)} + \&c\right\}$

$= \frac{1}{n} \cdot \frac{1}{n} + \frac{x^2}{n(n+1)(n+2)} \cdot \frac{1}{n+1} + \frac{x^4}{n(n+1)(n+2)(n+3)(n+4)} \cdot \frac{1}{n+2} + \&c$

9. $\left(1 + \frac{x}{1.3} + \frac{x^2}{1.3.5} + \frac{x^3}{1.3.5.7} + \&c\right)\left(1 - \frac{x}{1.3} + \frac{x^2}{1.3.5} - \frac{x^3}{1.3.5.7} + \&c\right)$

$= 1 + \frac{x^2}{1.3.5} \cdot \frac{1}{3} + \frac{x^4}{1.3.5.7.9} \cdot \frac{1}{5} + \&c$

11. $\left\{1 + xn + x^2 n(n-1) + x^3 n(n-1)(n-2) + \&c\right\}$

$\times \left\{1 - xn + x^2 n(n-1) - x^3 n(n-1)(n-2) + \&c\right\}$

$= \frac{n}{n} + \frac{x^2}{n-1} n(n-1)(n-2) + \frac{x^4}{n-2} n(n-1)(n-2)(n-3)(n-4) + \&c$

27. $1 + \frac{x}{\underline{|1}} \cdot \frac{y}{x+y+1} + \frac{x(x+1)}{\underline{|2}} \cdot \frac{y(y+1)}{(x+y+1)(x+y+3)} +$

$\frac{x(x+1)(x+2)}{\underline{|3}} \cdot \frac{y(y+1)(y+2)}{(x+y+1)(x+y+3)(x+y+5)} + \&c = \frac{\underline{|\frac{x+y-1}{2}}}{\underline{|\frac{x-1}{2}}\ \underline{|\frac{y-1}{2}}} \sqrt{\pi}$

Cor. 1. $\frac{\underline{|\frac{n-1}{2}}}{\underline{|\frac{n+x-2}{4}}\ \underline{|\frac{n-x-2}{4}}} \sqrt{\frac{2\pi}{2^n}} = 1 + \frac{1^2 - x^2}{4(n+1)} + \frac{(1^2-x^2)(3^2-x^2)}{4.8.(n+1)(n+3)}$

$+ \frac{(1^2-x^2)(3^2-x^2)(5^2-x^2)}{4.8.12.(n+1)(n+3)(n+5)} + \&c$

$$\int_0^\infty \frac{x^{n-1}}{1+x}\left\{1-\frac{\alpha\beta}{(\alpha+\beta)\underline{\lfloor 1}}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)\underline{\lfloor 2}}x^2-\&c\right\}dx$$

$$=\frac{\underline{\lfloor \alpha-n}\ \underline{\lfloor \beta-n}\ \underline{\lfloor n-1}}{\underline{\lfloor \alpha+\beta-n-1}}\left\{\frac{1}{\alpha+\beta-n}+\frac{\alpha\beta}{(\alpha+\beta)\underline{\lfloor 1}}\cdot\frac{1}{\alpha+\beta-n+1}\right.$$

$$\left.+\frac{\alpha(\alpha+1)\beta(\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)\underline{\lfloor 2}}\cdot\frac{1}{\alpha+\beta-n+2}+\&c\right\}$$

$$=\frac{\underline{\lfloor \alpha-n}\ \underline{\lfloor n-1}}{\underline{\lfloor \alpha-1}}\left\{\frac{1}{\alpha}+\frac{\alpha n}{(\alpha+\beta)\underline{\lfloor 1}}\cdot\frac{1}{\alpha+1}+\frac{\alpha(\alpha+1)n(n+1)}{(\alpha+\beta)(\alpha+\beta+1)\underline{\lfloor 2}}\right.$$

$$\left.\times\frac{1}{\alpha+2}+\&c\right\}.$$

If $\alpha+\beta=1$, then

$$\frac{1+\frac{\alpha}{\underline{\lfloor 1}}\cdot\frac{\beta}{\gamma+1}\cdot\left(\frac{1-\sqrt{1-x}}{2}\right)+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\underline{\lfloor 2}(\gamma+1)(\gamma+2)}\cdot\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c}{1+\frac{\alpha+\gamma}{4}\cdot\frac{\beta+\gamma}{\gamma+1}x+\frac{(\alpha+\gamma)(\alpha+\gamma+2)(\beta+\gamma)(\beta+\gamma+2)}{4\cdot 8\,(\gamma+1)(\gamma+2)}x^2+\&c}$$

$$=\left(\frac{1+\sqrt{1-x}}{2}\right)^{\gamma}.$$

from XIII 11 & 29 or 29 alone.

2. $\frac{\lfloor\frac{n-1}{2}}{\left(\lfloor\frac{n-2}{2}\right)^2}\sqrt{\frac{2\pi}{2n}} = 1 + \frac{1^2}{4(n+1)} + \frac{1^2\cdot 3^2}{4\cdot 8\cdot(n+1)(n+3)} + \&c$

3. $\frac{\lfloor\frac{n-1}{2}}{\lfloor\frac{2n-3}{8}\,\lfloor\frac{2n-5}{8}}\sqrt{\frac{2\pi}{2n}} = 1+\frac{1\cdot3}{16(n+1)}+\frac{1\cdot3\cdot5\cdot7}{16\cdot32(n+1)(n+3)}+\&c$

28. If $x+y+z=0$, then

$$\frac{1}{n} + \frac{x}{\lfloor 1}\cdot\frac{y}{z}\cdot\frac{1}{n+1} + \frac{x(x-1)}{\lfloor 2}\cdot\frac{y(y-1)}{z(z+1)}\cdot\frac{1}{n+2}+\&c$$

$$= \frac{\lfloor n-1\,\lfloor x+y+n}{\lfloor x+n\,\lfloor y+n}\left\{1+\frac{x}{\lfloor 1}\cdot\frac{y}{z}+\frac{x(x-1)}{\lfloor 2}\cdot\frac{y(y-1)}{z(z+1)}+\&c \text{ to } x+y+n+1 \text{ terms}\right\}$$

29. If $x+y+z = \frac{1}{2}$, then

$$1+\frac{x}{\lfloor 1}\cdot\frac{y}{z}p + \frac{x(x-1)}{\lfloor 2}\cdot\frac{y(y-1)}{z(z+1)}p^2+\frac{x(x-1)(x-2)}{\lfloor 3}\cdot\frac{y(y-1)(y-2)}{z(z+1)(z+2)}p^3+\&c$$

$$=1+\frac{2x}{\lfloor 1}\cdot\frac{2y}{z}\cdot\left(\frac{1-\sqrt{1-p}}{2}\right)+\frac{2x(2x-1)}{\lfloor 2}\cdot\frac{2y(2y-1)}{z(z+1)}\cdot\left(\frac{1-\sqrt{1-p}}{2}\right)^2+\&c$$

Cor. $1+\frac{1^2+n}{4^2}x+\frac{(1^2+n)(5^2+n)}{4^2\cdot8^2}x^2+\frac{(1^2+n)(5^2+n)(9^2+n)}{4^2\cdot8^2\cdot12^2}x^3+\&c$

$$=1+\frac{1^2+n}{2^2}\left(\frac{1-\sqrt{1-x}}{2}\right)+\frac{(1^2+n)(3^2+n)}{2^2\cdot4^2}\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c$$

30. If $x+y+z=0$, then

$$\left\{1+\frac{x}{\lfloor 1}\cdot\frac{y}{z+\frac{1}{2}}p+\frac{x(x-1)}{\lfloor 2}\cdot\frac{y(y-1)}{(z+\frac{1}{2})(z+1\frac{1}{2})}p^2+\&c\right\}^2$$

$$=1+\frac{2x}{\lfloor 1}\cdot\frac{2y}{z+\frac{1}{2}}\cdot\frac{z}{2z}p+\frac{2x(2x-1)}{\lfloor 2}\cdot\frac{2y(2y-1)}{(z+\frac{1}{2})(z+1\frac{1}{2})}p^2+\&c$$

Cor 1. $\left\{1+\frac{1^2+n}{4^2}x+\frac{(1^2+n)(5^2+n)}{4^2\cdot8^2}x^2+\&c\right\}^2$

$$=1+\frac{1}{2}\cdot\frac{1^2+n}{2^2}x+\frac{1\cdot3}{2\cdot4}\cdot\frac{(1^2+n)(3^2+n)}{2^2\cdot4^2}x^2+\&c$$

2. $\left(1+\frac{x}{2^2}+\frac{x^2}{2^2\cdot4^2}+\frac{x^3}{2^2\cdot4^2\cdot6^2}+\&c\right)^2$

$$=1+\frac{1}{2}\cdot\frac{x}{(\lfloor 1)^2}+\frac{1\cdot3}{2\cdot4}\cdot\frac{x^2}{(\lfloor 2)^2}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^3}{(\lfloor 3)^3}+\&c$$

* If pth & qth be $\phi(x)$ & $\psi(x)$ and nth be $f(x)$

then (i) if pth & qth be $\phi F(x)$ & $\psi F(x)$ then nth $= f F(x)$.

(ii) if pth & qth be $F\phi(x)$ & $F\psi(x)$ then nth $= F f(x)$.

Thus we may add or subtract any constant (& multiply or divide by any constant) to x in each function or to each function

$I = x \quad II = x^2 + 2x \quad$ then nth $= (x+1)^n - 1$

$I = x \quad II = x^2 + 4x \quad$ then nth $= \left\{\left(\dfrac{\sqrt{x+4}+\sqrt{x}}{2}\right)^n - \left(\dfrac{\sqrt{x+4}-\sqrt{x}}{2}\right)^n\right\}^2$

$I = x \quad II = x^2 - 2 \quad$ then nth $= \left(\dfrac{x+\sqrt{x^2-4}}{2}\right)^n + \left(\dfrac{x-\sqrt{x^2-4}}{2}\right)^n$

$I = x \quad II = \dfrac{x^2}{(2-x)^2} \quad n$th $= \dfrac{4x^n}{\left\{(1+\sqrt{1-x})^n + (1-\sqrt{1-x})^n\right\}^2}$

If $I = x$ & $II\ x^2 + 2nx$, then

$III = x^3 + 3nx^2 + 3.\dfrac{n(n+1)}{2}x - \dfrac{n(n-1)(n-2)x}{2x + \dfrac{3(n+1)}{2}}$.

If $\chi(x)$ y be the common equation then the required series $= \dfrac{y}{x(1-x)\,\partial y/\partial x}$ or any power of this or more generally if $f(x)$ & $F(x)$ be of the pth & qth degree find $\phi(x)$ such that $\sqrt[p]{\phi f(x)} = \sqrt[q]{\phi F(x)} = \chi(x)$, then

CHAPTER XIV

1. $\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots+\frac{a_r}{b_r} = a_1\frac{N_{r-1}}{D_r} = \frac{a_1}{D_0 D_1} - \frac{a_1 a_2}{D_1 D_2} + \frac{a_1 a_2 a_3}{D_2 D_3} - \&c$ to r terms.

$N_{r-1} = b_r N_{r-2} + a_r N_{r-3}$ and $D_r = b_r D_{r-1} + a_r D_{r-2}$.

Cor. $a_1+a_2+a_3+\&c$ to r terms $= \frac{a_1}{1-}\frac{a_2}{a_1+a_2-}\frac{a_1 a_3}{a_2+a_3-}\frac{a_2 a_4}{a_3+a_4-}\frac{a_3 a_5}{a_4+a_5-}\&c$ to r terms.

2. $x = x - a_1 + \frac{x a_1}{x-a_2+}\frac{x a_2}{x-a_3+}\frac{x a_3}{x-a_4+}\&c$

3. $x = a_1 + \sqrt{x^2 + a_1(a_1+2a_2) - 2a_1\sqrt{x^2 + a_2(a_2+2a_3) - 2a_2\sqrt{\&c\ \&c}}}$

4. $x+n+a = \sqrt{ax+(n+a)^2 + x\sqrt{a(x+n)+(n+a)^2+(x+n)\sqrt{\&c}}}$

Ex. 1. $3 = 1\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{\&c}}}}$

2. $4 = 1\sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\&c}}}}$

5. $\frac{1}{x+a} - \frac{1}{(x+a)(x+2a)} + \frac{1}{(x+a)(x+2a)(x+3a)} - \&c$

$= \frac{1}{x+a+}\frac{x+a}{x+2a-1+}\frac{x+2a}{x+3a-1+}\frac{x+3a}{x+4a-1+}\&c$

Cor. $\frac{1}{e-1} = \frac{1}{1+}\frac{2}{2+}\frac{3}{3+}\&c$

6. $x = \frac{x+1}{x+}\frac{x+2}{x+1+}\frac{x+3}{x+2+}\&c$

Cor. $1 = \frac{2}{1+}\frac{3}{2+}\frac{4}{3+}\&c$

7. $\frac{x+a+1}{x+1} = \frac{x+2}{x-1+}$ [illegible] $\frac{}{x+a-1+}\frac{x+3a}{x+2a-1}$ [illegible]

the function for the nth degree $= \phi^{-1}\{\chi(x)\}^n$,

The self-repeating series ~~=~~ $\sqrt[n]{\dfrac{\phi(x)}{\psi(x)\phi'(x)}}$ where n is any quantity and $\psi(x)$ any known functions. supposing the series to be ~~equal~~ $S(x)$. then

$$\frac{S\,F(x)}{S\,f(x)} = \sqrt[n]{\frac{p}{q}\cdot\frac{\psi f(x)\,F'(x)}{\psi F(x)\,f'(x)}}$$

If $y = e^{-2\pi \cdot \frac{1+\frac{1.5}{6^2}(1-x)+\&c}{1+\frac{1.5}{6^2}x+\&c}}$

then $1 - 504\left(\frac{1^5 y}{1-y} + \frac{2^5 y^2}{1-y^2} + \frac{3^5 y^3}{1-y^3} + \&c\right)$

$= \left\{1 + \frac{1.5}{6^2}x + \&c\right\}^6 (1-2x)$.

If $y = e^{-\frac{2\pi}{\sqrt{3}} \cdot \frac{1+\frac{1.2}{3^2}(1-x)+\&c}{1+\frac{1.2}{3^2}x+\&c}}$

then $1 + 240\left(\frac{1^3 y}{1-y} + \frac{2^3 y^2}{1-y^2} + \frac{3^3 y^3}{1-y^3} + \&c\right)$

$= \left\{1 + \frac{1.2}{3^2}x + \&c\right\}^4 (1+8x)$

$1 - 504\left(\frac{1^5 y}{1-y} + \frac{2^5 y^2}{1-y^2} + \frac{3^5 y^3}{1-y^3} + \&c\right)$

$= \left\{1 + \frac{1.2}{3^2}x + \&c\right\}^6 (1 - 20x - 8x^2)$

If $y = e^{-\pi\sqrt{2} \cdot \frac{1+\frac{1.3}{4^2}(1-x)+\&c}{1+\frac{1.3}{4^2}x+\&c}}$ then

$1 + 240\left(\frac{1^3 y}{1-y} + \frac{2^3 y^2}{1-y^2} + \&c\right) = \left(1 + \frac{1.3}{4^2}x + \&c\right)^4 (1+3x)$.

$1 - 504\left(\frac{1^5 y}{1-y} + \frac{2^5 y^2}{1-y^2} + \&c\right) = \left(1 + \frac{1.3}{4^2}x + \&c\right)^6 (1-9x)$.

Cor. 1. $\frac{4}{3} = \frac{3}{1+}\frac{4}{2+}\frac{5}{3+}\frac{6}{4+}$ &c

2. $\frac{5}{3} = \frac{4}{1+}\frac{6}{3+}\frac{8}{5+}\frac{10}{7+}$ &c

8. If x is a positive integer,

$$x = \frac{1}{1-x+}\frac{2}{2-x+}\frac{3}{3-x+}\cdots+\frac{x}{0+}\frac{x+1}{1+}\frac{x+2}{2+}\text{ \&c}$$

9. If a is a positive integer and if

$$\frac{N_a}{D_a} = \frac{n}{n-a+}\frac{n+1}{n-a+1+}\frac{n+2}{n-a+2+}\text{ \&c},$$ then

$$\frac{N_{a+1}}{N_a} = n+2-a+\frac{a-1}{n+3-a+}\frac{a-2}{n+4-a+}\text{ \&c}$$

Here we should equate the numerators and the denominators (in lowest terms)

If $N = \phi(n)$, then $D = \phi(n-1)$.

Cor 1. $\frac{n^2+n+1}{n^2-n+1} = \frac{n}{n-3+}\frac{n+1}{n-2+}\frac{n+2}{n-1+}$ &c

2. $\frac{n^3+2n+1}{(n-1)^3+2(n-1)+1} = \frac{n}{n-4+}\frac{n+1}{n-3+}\frac{n+2}{n-2+}$ &c

10. $1 = \frac{x+a}{a+}\frac{(x+a)^2-a^2}{a+}\frac{(x+2a)^2-a^2}{a+}$ &c

11. If $a < b$, then

$$a = \frac{ab}{a+b+d-}\frac{(a+d)(b+d)}{a+b+3d-}\frac{(a+2d)(b+2d)}{a+b+5d-}\text{ [illegible]}$$

12. $\frac{a_1}{x+}\frac{a_2}{1+}\frac{a_3}{x+}\frac{a_4}{1+}$ &c to $2n$ terms

$$= \frac{a_1}{x+a_2-}\frac{a_2 a_3}{x+a_3+a_4-}\frac{a_4 a_5}{x+\text{[illegible]}}\text{ [illegible] -- \&c [illegible]}$$

$$\sqrt{2}\left\{\frac{1}{2}+e^{-\frac{\pi x}{x^2+y^2}}\cos\left(\frac{\pi y}{x^2+y^2}\right)+e^{-\frac{4\pi x}{x^2+y^2}}\cos\left(\frac{4\pi y}{x^2+y^2}\right)+\&c\right\}$$

$$=\sqrt{\sqrt{x^2+y^2}+x}\left\{\frac{1}{2}+e^{-\pi x}\cos\pi y+e^{-4\pi x}\cos 4\pi y+\&c\right\}$$

$$+\sqrt{\sqrt{x^2+y^2}-x}\left\{e^{-\pi x}\sin\pi y+e^{-4\pi x}\sin 4\pi y+\&c\right\}$$

$$\sqrt{2}\left\{e^{-\frac{\pi x}{x^2+y^2}}\sin\left(\frac{\pi y}{x^2+y^2}\right)+e^{-\frac{4\pi x}{x^2+y^2}}\sin\left(\frac{4\pi y}{x^2+y^2}\right)+\&c\right\}$$

$$=\sqrt{\sqrt{x^2+y^2}-x}\left\{\frac{1}{2}+e^{-\pi x}\cos\pi y+e^{-4\pi x}\cos 4\pi y+\&c\right\}$$

$$-\sqrt{\sqrt{x^2+y^2}+x}\left\{e^{-\pi x}\sin\pi y+e^{-4\pi x}\sin 4\pi y+\&c\right\}$$

$$\frac{1}{1^2}-\frac{4}{3^2}+\frac{4}{5^2}-\&c=.915\,965,\,594\,177$$

If $\alpha \equiv \dfrac{27p(1+p)^4}{2(1+4p+p^2)^3}$ & $\beta=\dfrac{27p^4(1+p)}{2(2+2p-p^2)^3}$ then

$$\left(1+p-\frac{p^2}{2}\right)\left\{1+\frac{1.2}{3^2}\alpha+\frac{1.2.4.5}{3^2.6^2}\alpha^2+\&c\right\}$$

$$=\left(1+4p+p^2\right)\left\{1+\frac{1.2}{3^2}\beta+\frac{1.2.4.5}{3^2.6^2}\beta^2+\&c\right\}$$

13. $$\cfrac{a_1+h}{1+\cfrac{a_1}{x+\cfrac{a_2+h}{1+\cfrac{a_2}{x+\cfrac{a_3+h}{1+\&c}}}}}$$

$$= h+\cfrac{a_1}{1+\cfrac{a_1+h}{x+\cfrac{a_2}{1+\cfrac{a_2+h}{x+\&c}}}}$$

14. $$\frac{1}{(m+1)(n+1)}-\frac{1}{(m+2)(n+2)}+\&c$$

$$=\cfrac{1}{m+n+1+mn+\cfrac{(m+1)^2(n+1)^2}{m+n+3+\cfrac{(m+2)^2(n+2)^2}{m+n+5+\&c}}}$$

15. $$\cfrac{a_1x}{b_1+\cfrac{a_2x}{b_2+\cfrac{a_3x}{b_3+\&c}}} = T_1x-T_2x^2+T_3x^3-\&c$$

Let $\frac{P_n}{b_n}=\frac{a_1a_2\cdots a_n}{(b_1b_2\cdots b_n)^2}$ and $T_n-P_n=t_n$, then

$t_1=0,\ t_2=0$

$T_1t_3-T_2^2=0$

$T_2t_4-T_3^2=0$

$T_3t_5-T_4^2=\frac{M^2}{P_1P_3}$ where $M=T_2T_4-T_3^2$

$T_4t_6-T_5^2=\frac{N^2}{P_2P_4}$ where $N=T_3T_5-T_4^2$

16. $$\frac{(x+1)^n-(x-1)^n}{(x+1)^n+(x-1)^n}=\cfrac{n}{x+\cfrac{n^2-1^2}{3x+\cfrac{n^2-2^2}{5x+\cfrac{n^2-3^2}{7x+\&c}}}} = A_n.$$

$$A_n+\frac{1}{A_n}=\frac{2}{A_{2n}}$$

Cor 1. $$\tan^{-1}\frac{1}{x}=\cfrac{1}{x+\cfrac{1^2}{3x+\cfrac{2^2}{5x+\cfrac{3^2}{7x+\&c}}}}$$

2. $$\log_e\frac{x+1}{x-1}=\cfrac{2}{x-\cfrac{1^2}{3x-\cfrac{2^2}{5x-\cfrac{3^2}{7x-\&c}}}}$$

3. $$\tan\frac{1}{x}=\cfrac{1}{x-\cfrac{1}{3x-\cfrac{1}{5x-\cfrac{1}{7x-\&c}}}}$$

4. $$\frac{e^x-1}{e^x+1}=\cfrac{x}{2+\cdots}$$

$$e^{-\pi\sqrt{2}} \cdot \frac{1 + \frac{1.3}{4^2}(1-x) - \cdots}{1 + \frac{1.3}{4^2} x + } = y$$

III degree $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 4\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)} = 1$

VII degree $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 20\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}$

$$+ 8\sqrt{2}\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\left\{\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right\} = 1$$

$$\left\{1 + \frac{1.3}{4^2}x + \frac{1.3.5.7}{4^2.8^2}x^2 + \&c\right\}^2$$

$$= 1 + 24\left(\frac{y}{1-y} + \frac{3y^3}{1-y^3} + \frac{5y^5}{1-y^5} + \&c\right)$$

V degree $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 8\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}\left(\sqrt[6]{\alpha\beta} + \sqrt[6]{(1-\alpha)(1-\beta)}\right)$

$$= 1$$

$$F\left(\frac{3\sqrt{2} - \sqrt{5} - 2}{3\sqrt{2} + \sqrt{5} + 2}\right) = e^{-\pi\sqrt{10}} = F\left\{(\sqrt{10} - 3)^2(\sqrt{2} - 1)^4\right\}.$$

$$F\left(\frac{7\sqrt{2} - 2\sqrt{6} - 5}{7\sqrt{2} + 2\sqrt{6} + 5}\right) = e^{-3\pi\sqrt{2}}$$

$$1 + \frac{1.2}{3^2}\left\{1 - \left(\frac{1-t}{1+2t}\right)^3\right\} + \&c$$

$$= (1 + 2t)\left\{1 + \frac{1.2}{3^2}t^3 + \frac{1.2.4.5}{3^2.6^2}t^6 + \&c\right\}$$

XI $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 16\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}\left(\sqrt[3]{\alpha\beta} + \sqrt[3]{1-\alpha.1-\beta}\right)$

$$+ 48\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}\left(\sqrt[6]{\alpha\beta} + \sqrt[6]{(1-\alpha)(1-\beta)}\right) = 1$$

$$+ 68\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}$$

17. $$\frac{\frac{x}{n} + \frac{x^2}{\lfloor 1}\cdot\frac{1}{n(n+1)} + \frac{x^3}{\lfloor 2}\cdot\frac{1}{n(n+1)(n+2)} + \&c}{1 + \frac{x}{\lfloor 1}\cdot\frac{1}{n} + \frac{x^2}{\lfloor 2}\cdot\frac{1}{n(n+1)} + \&c}$$

$$= \cfrac{x}{n + \cfrac{x}{n+1 + \cfrac{x}{n+2 + \cfrac{x}{n+3 + \&c}}}}$$

Sol: Let $\phi(x) = \frac{x}{\lfloor 0 \lfloor 1} + \frac{x^2}{\lfloor 1 \lfloor 2} + \frac{x^3}{\lfloor 2 \lfloor 3} + \frac{x^4}{\lfloor 3 \lfloor 4} + \&c$, then we see tha[t]

$$\phi(x) = x\phi''(x)$$

$$\therefore \phi'(x) = \phi''(x) + x\phi'''(x)$$

$$\therefore \phi''(x) = 2\phi'''(x) + x\phi^{IV}(x)$$

$$\phi'''(x) = 3\phi^{IV}(x) + x\phi^{V}(x)$$

&c &c

$$\therefore \frac{\phi(x)}{\phi'(x)} = \frac{x\phi''(x)}{\phi''(x) + x\phi'''(x)} = \cfrac{x}{1 + x\cfrac{\phi'''(x)}{\phi''(x)}} = \cfrac{x}{1 + \cfrac{x\phi''(x)}{2\phi'''(x) + x\phi^{IV}(x)}}$$

$$= \cfrac{x}{1 + \cfrac{x}{2 + x\cfrac{\phi^{IV}(x)}{\phi'''(x)}}}$$

$$= \cfrac{x}{1 + \cfrac{x}{2 + \cfrac{x}{3 + \cdots + \cfrac{x}{n-1 + \cfrac{\frac{x}{n} + \frac{x^2}{\lfloor 1}\cdot\frac{1}{n(n+1)} + \frac{x^3}{\lfloor 2}[illegible]}{1 + \frac{x}{\lfloor 1}[illegible]}}}}}$$

18. $$\frac{\frac{m}{n}x + \frac{r-n}{\lfloor 1}\cdot\frac{m(m+1)}{n(n+1)}x^2 + \frac{(r-n)(r-n-1)}{\lfloor 2}\cdot\frac{m(m+1)(m+2)}{n(n+1)(n+2)}[illegible]}{1 + \frac{r-n}{\lfloor 1}\cdot\frac{m}{n}x + \frac{(r-n)(r-n-1)}{\lfloor 2}\cdot\frac{m(m+1)}{n(n+1)}[illegible]}$$

$$= \cfrac{xmr}{n + \cfrac{x(m-n)(r-n)}{n+1 + \cfrac{x(m+1)(r+1)}{n+2 + \cfrac{x(m-n-1)(r-n-1)}{n+3 + \&c}}}}$$

19. $$\frac{n}{m}x + \frac{n(n+1)}{m(m+1)}x^2 + \&c$$

$$= \cfrac{nx}{m + \cfrac{m(n+1)x}{m+1 + \cfrac{1(m-n)x}{m+2 + \cfrac{(m+1)(n+2)x}{m+3 + [illegible]}}}}$$

$$= \frac{nx}{m + [illegible]} \quad 1 + \frac{1(1+x)}{m + \frac{(n+2)x}{1 + [illegible]}}$$

$$= \cfrac{nx}{m + x(n+1) - \cfrac{1(n+1)x(1+x)}{m+1 + x(n+2) - \cfrac{2(n+2)x(1+x)}{m+2 + x[illegible]}}}$$

If $y = e^{-2\pi \cdot \frac{1 + \frac{1.5}{6^2}(1-x) + \frac{1.5.7.11}{6^2.12^2}(1-x)^2 + \&c}{1 + \frac{1.5}{6^2}x + \frac{1.5.7.11}{6^2.12^2}x^2 + \&c}}$

then $\left\{1 + \frac{1.5}{6^2}x + \frac{1.5.7.11}{6^2.12^2}x^2 + \&c\right\}^4$

$= 1 + 240\left(\frac{1^3 y}{1-y} + \frac{2^3 y^2}{1-y^2} + \frac{3^3 y^3}{1-y^3} + \&c\right)$

If $1 + \frac{1.5}{6^2}x + \frac{1.5.7.11}{6^2.12^2}x^2 + \&c$

$= \left\{1 + \left(\frac{1}{2}\right)^2 y + \left(\frac{1.3}{2.4}\right)^2 y^2 + \&c\right\}\sqrt[4]{1-y+y^2}$

& $x(1-x) = A$ & $y(1-y) = B$

then $A = \frac{27B^2}{16(1-B)^3}$

If $y = \frac{p(2+p)}{1+2p}$ then $x = \frac{27}{4} \cdot \frac{(p+p^2)^2}{(1+p+p^2)^3}$

If $z = \frac{p^3(2+p)}{1+2p}$ and $x = \frac{27}{4} \cdot \frac{(p+p^2)^2}{(1+p+p^2)^3}$

then $1 + \frac{1.2}{3^2}x + \frac{1.2.4.5}{3^2.6^2}x^2 + \&c$

$= \left\{1 + \left(\frac{1}{2}\right)^2 z + \left(\frac{1.3}{2.4}\right)^2 z^2 + \&c\right\}\frac{1+p+p^2}{\sqrt{1+2p}}$

Sol. Write $n+1$ for n in XIV 18. Subtract both sides from 1 and equate the reciprocals of the result.

Cor 1. $\frac{x}{m} + \frac{x^2}{m(m+1)} + \frac{x^3}{m(m+1)(m+2)} + \&c$

$$= \cfrac{x}{m - \cfrac{mx}{m+1 + \cfrac{1x}{m+2 - \cfrac{(m+1)x}{m+3 + \cfrac{2x}{m+4 - \&c}}}}}$$

$$= \cfrac{x}{m-x + \cfrac{x}{m+1-x + \cfrac{2x}{m+2-x + \cfrac{3x}{m+3-x + \&c}}}}$$

Cor 2. $1 + \frac{x}{x+1} + \frac{x^2}{(x+1)(x+2)} + \&c$

$$= 1 + \cfrac{2x}{2 + \cfrac{3x}{3 + \cfrac{4x}{4 + \&c}}}$$

20.
$$\frac{\underline{\left|\frac{x+m+n-1}{2}\right.}\;\underline{\left|\frac{x-m-n-1}{2}\right.} - \underline{\left|\frac{x+m-n-1}{2}\right.}\;\underline{\left|\frac{x-m+n-1}{2}\right.}}{\underline{\left|\frac{x+m+n-1}{2}\right.}\;\underline{\left|\frac{x-m-n-1}{2}\right.} + \underline{\left|\frac{x+m-n-1}{2}\right.}\;\underline{\left|\frac{x-m+n-1}{2}\right.}}$$

$$= \cfrac{mn}{x + \cfrac{(m^2-1^2)(n^2-1^2)}{3x + \cfrac{(m^2-2^2)(n^2-2^2)}{5x + \cfrac{(m^2-3^2)(n^2-3^2)}{7x + \&c}}}}$$

21.
$$\frac{\underline{\left|\frac{x+n-3}{4}\right.}\;\underline{\left|\frac{x-n-3}{4}\right.}}{\underline{\left|\frac{x+n-1}{4}\right.}\;\underline{\left|\frac{x-n-1}{4}\right.}} = \cfrac{4}{x - \cfrac{n^2-1^2}{2x - \cfrac{n^2-3^2}{2x - \cfrac{n^2-5^2}{2x - \&c}}}}$$

Cor 1.
$$\left(\frac{\underline{\left|\frac{x-3}{4}\right.}}{\underline{\left|\frac{x-1}{4}\right.}}\right)^2 = \cfrac{4}{x + \cfrac{1^2}{2x + \cfrac{3^2}{2x + \cfrac{5^2}{2x + \&c}}}}$$

Cor 2.
$$\frac{\underline{\left|\frac{x-5}{8}\right.}\;\underline{\left|\frac{x-7}{8}\right.}}{\underline{\left|\frac{x-1}{8}\right.}\;\underline{\left|\frac{x-3}{8}\right.}} = \cfrac{8}{x + \cfrac{1.3}{2x + \cfrac{5.7}{2x + \cfrac{9.11}{2x + \&c}}}}$$

22.
$$\left\{\frac{\underline{\left|\frac{x+n-3}{4}\right.}\;\underline{\left|\frac{x-n-3}{4}\right.}}{\underline{\left|\frac{x+n-1}{4}\right.}\;\underline{\left|\frac{x-n-1}{4}\right.}}\right\}^2 = \cfrac{8}{\frac{x^2+n^2-1}{2} + \cdots}$$

$\times$ [illegible]

$$= \cdots$$

$$\underline{\left|\frac{x-3}{4}\right.} \; [illegible] = \frac{8}{x \cdots}$$

$$\alpha = \frac{p(3+p)^2}{2(1+p)^3}; \quad \beta = \frac{p^2(3+p)}{4}$$

$$1-\alpha = \frac{(1-p)^2(2+p)}{2(1+p)^3}, \quad 1-\beta = \frac{(1-p)(2+p)^2}{4}.$$

$$1 + \frac{1.2}{3^2}\alpha + \frac{1.2.4.5}{3^2.6^2}\alpha^2 + \frac{1.2.4.5.7.8}{3^2.6^2.9^2}\alpha^3 + \&c$$

$$= (1+p)\left\{1 + \frac{1.2}{3^2}\beta + \frac{1.2.4.5}{3^2.6^2}\beta^2 + \frac{1.2.4.5.7.8}{3^2.6^2.9^2}\beta^3 + \&c\right\}$$

II deg. $\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} = 1$

$$\sqrt[3]{\frac{\alpha^2}{\beta}} - \sqrt[3]{\frac{(1-\alpha)^2}{1-\beta}} = \frac{2}{1+p}$$

$$\sqrt[3]{\frac{(1-\beta)^2}{1-\alpha}} - \sqrt[3]{\frac{\beta^2}{\alpha}} = 1+p.$$

$$\sqrt[3]{\frac{\alpha^2}{\beta}} + \sqrt[3]{\frac{(1-\alpha)^2}{1-\beta}} = \frac{4}{(1+p)^2}$$

$$\sqrt[3]{\frac{(1-\beta)^2}{1-\alpha}} + \sqrt[3]{\frac{\beta^2}{\alpha}} = (1+p)^2$$

V deg. $\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} + 3\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)} = 1$

XI. $\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} + 3\sqrt{3}\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}\left\{\sqrt[6]{\alpha\beta} + \sqrt[6]{(1-\alpha)(1-\beta)}\right\}$
$+ 6\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)} = 1.$

1, 2, 4, 8

$\sqrt[3]{\alpha\delta} + \sqrt[3]{(1-\alpha)(1-\delta)} + 3\sqrt[6]{\beta\gamma(1-\beta)(1-\gamma)}$ ×

$$\frac{1 + \frac{1.2}{3^2}\beta + \&c}{1 + \frac{1.2}{3^2}\alpha + \&c}, \quad \frac{1 + \frac{1.2}{3^2}\gamma + }{1 + \frac{1.2}{3^2}\delta + } = 1$$

23. $$\left(\frac{1}{x+n+1} - \frac{1}{x+n+3} + \frac{1}{x+n+5} - \frac{1}{x+n+7} + \&c\right)$$
$$+\left(\frac{1}{x-n+1} - \frac{1}{x-n+3} + \frac{1}{x-n+5} - \frac{1}{x-n+7} + \&c\right)$$
$$= \cfrac{1}{x + \cfrac{1^2-n^2}{x + \cfrac{2^2}{x + \cfrac{3^2-n^2}{x + \cfrac{4^2}{x + \&c}}}}}$$

Cor. $$2\left(\frac{1}{x+1} - \frac{1}{x+3} + \frac{1}{x+5} - \frac{1}{x+7} + \&c\right) = \cfrac{1}{x + \cfrac{1^2}{x + \cfrac{2^2}{x + \cfrac{3^2}{x + \&c}}}}$$

24. $$\left(\frac{1}{x-n+1} + \frac{1}{x-n+3} + \frac{1}{x-n+5} + \&c\right)$$
$$-\left(\frac{1}{x+n+1} + \frac{1}{x+n+3} + \frac{1}{x+n+5} + \&c\right)$$
$$= \cfrac{n}{x + \cfrac{1^2(1^2-n^2)}{3x + \cfrac{2^2(2^2-n^2)}{5x + \cfrac{3^2(3^2-n^2)}{7x + \&c}}}}$$

Cor. $$2\left\{\frac{1}{(x+1)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+5)^2} + \&c\right\}$$
$$= \cfrac{1}{x + \cfrac{1^4}{3x + \cfrac{2^4}{5x + \cfrac{3^4}{7x + \&c}}}}$$

25. $$2x^2\left\{\frac{1}{2x^2} - \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} - \frac{1}{(x+3)^2} + \&c\right\}$$
$$= \cfrac{1}{x + \cfrac{1^2}{x + \cfrac{1.2}{x + \cfrac{2^2}{x + \cfrac{2.3}{x + \cfrac{3^2}{x + \&c}}}}}}$$

26. $$2x\left(\frac{1}{2x} - \frac{1}{x+2} + \frac{1}{x+4} - \frac{1}{x+6} + \&c\right)$$
$$= \cfrac{1}{x + \cfrac{1.2}{x + \cfrac{2.3}{x + \cfrac{3.4}{x + \cfrac{4.5}{x + \&c}}}}}$$

27. $$\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \frac{1}{(x+4)^3} + \&c$$
$$= \frac{1}{2x^2+2x} + \cfrac{1^3}{1 + \cfrac{1^3}{6x^2+6x} + \cfrac{2^3}{1 + \cfrac{2^3}{10x^2+10x} + \cdots}}$$

[illegible] $$\frac{2^6}{5(2x^2+2x+3)}$$ [illegible]

If $y = e^{-\frac{2\pi}{\sqrt{3}} \cdot \frac{1+\frac{1.2}{3^2}(1-x)+\frac{1.2.4.5}{3^2.6^2}(1-x)^2+\&c}{1+\frac{1.2}{3^2}x+\frac{1.2.4.5}{3^2.6^2}x^2+\&c}}$

then $1+\frac{1.2}{3^2}x+\frac{1.2.4.5}{3^2.6^2}x^2+\&c$

$$=1+6\left(\frac{y}{1-y}-\frac{y^2}{1-y^2}+\frac{y^4}{1-y^4}-\frac{y^5}{1-y^5}+\&c\right)$$

$$1+12\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{4x^4}{1-x^4}+\frac{5x^5}{1-x^5}+\&c\right)$$

$$=\left\{1+6\left(\frac{x}{1-x}-\frac{x^2}{1-x^2}+\frac{x^4}{1-x^4}-\frac{x^5}{1-x^5}+\&c\right)\right\}^2$$

$$1+4\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\frac{4x^4}{1-x^4}+\frac{5x^5}{1-x^5}+\frac{6x^6}{1-x^6}+\frac{8x^8}{1-x^8}\right.$$

$$=\left\{1+2\left(\frac{x}{1-x}+\frac{x^2}{1-x^2}-\frac{x^3}{1-x^3}+\frac{x^4}{1-x^4}-\frac{x^5}{1-x^5}-\frac{x^6}{1-x^6}\right.\right.$$

$$\left.\left.+\frac{x^8}{1-x^8}+\frac{x^9}{1-x^9}-\frac{x^{10}}{1-x^{10}}+\frac{x^{11}}{1-x^{11}}-\&c\right)\right\}^2$$

$$1+6\left(\frac{x}{1-x}-\frac{x^2}{1-x^2}+\frac{x^4}{1-x^4}-\&c\right)$$

$$=4\,\frac{\psi^3(x^2)}{\psi(x^6)}-3\,\frac{\phi^3(x^3)}{\phi(x)}$$

$$1+6\left(\frac{1}{e^y-1}-\frac{1}{e^{2y}-1}+\frac{1}{e^{4y}-1}-\&c\right)$$

$$=\frac{\phi^3(e^{-3y})}{\phi(e^{-y})}(1+4p+p^2)$$

Cor. $\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \&c$

$$= \frac{1}{2(x^2+x)+1} + \frac{1}{24(x^2+x)^3 + 60(x^2+x)^2 + \frac{72}{5}(x^2+x) + \frac{\phi(x)}{40}}$$

$\phi(0) = 198$, $\phi(\tfrac{1}{2}) = 571$, $\phi(1) = 1015$, $\phi(1\tfrac{1}{2}) = 1384$, $\phi(2) = 1679$, $\phi(2\tfrac{1}{2}) = 1916$, $\phi(3) = 2093$ nearly and $\phi(\infty) = 2880$.

If h is a positive proper fraction, then

$$\frac{\phi(2+h) - \phi(2)}{\phi(3) - \phi(2)} = \frac{3h\,\phi(2)}{2\phi(3) + \frac{h}{2}\{3\phi(2) - 2\phi(3)\}} \text{ nearly}$$

28. A series of the form $A_0 + (A_1 + A_{-1}) + (A_2 + A_{-2}) + \&c$ is called a perfect series. Hence we see that the series $A_0 + A_1 + A_2 + A_3 + \&c$ is only perfect when A_{-1}, A_{-2}, A_{-3} are all equal to 0. Thus $1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \&c$ is perfect.

If $\phi(0) + \phi(1) + \phi(2) + \phi(3) + \&c$ is a perfect series, then

$$\left.\begin{aligned} &\phi(x) + \{\phi(x+1) + \phi(x-1)\} + \{\phi(x+2) + \phi(x-2)\} + \&c \\ &= \phi(y) + \{\phi(y+1) + \phi(y-1)\} + \{\phi(y+2) + \phi(y-2)\} + \&c \end{aligned}\right\} \text{for all values of } x \text{ and } y$$

29. $A_0 + A_1 + A_2 + \cdots + A_n$

$$= (A_n + A_{n-1} + A_{n-2} + \&c \text{ ad inf.})$$
$$- (A_{-1} + A_{-2} + A_{-3} + \&c \text{ ad inf.}) \text{ for all values of } n$$

N. B. We also know that $A_0 + A_1 + \cdots A_n$ for all values of n

$$= (A_0 - A_{n+1}) + (A_1 - A_{n+2}) + (A_2 - A_{n+3}) + \&c \text{ [illegible]}$$

Cor. $\phi(0) + \frac{r}{\lfloor 1}\phi(1) + \frac{r(r-1)}{\lfloor 2}\phi(2) + \&c$ ad inf.

$$= \phi(r) + \frac{r}{\lfloor 1}\phi(r-1) + \frac{r(r-1)}{\lfloor 2}\phi(r-2) + \&c \text{ ad inf.}$$

Ex. If either or both of m & x be very great show that

$$1 + \frac{m}{n+1}x + \frac{m(m-1)}{(n+1)(n+2)}x^2 + \&c = \frac{\lfloor m \lfloor n}{[illegible]}(1+\theta)$$

where θ is less than $\frac{m}{x(m+1)}$ and shew that for all

$$x\psi^3(x)\psi(x^5) - 5x^2\psi(x)\psi^3(x^5)$$

$$= \frac{x}{1-x^2} - \frac{2x^2}{1-x^4} - \frac{3x^3}{1-x^6} + \frac{4x^4}{1-x^8} + \frac{6x^6}{1-x^{12}} \cdots$$

$$5\phi(x)\phi^3(x^5) - \phi^3(x)\phi(x^5)$$

$$= 4\left\{1 + \frac{x}{1+x} - \frac{2x^2}{1-x^2} - \frac{3x^3}{1+x^3} + \frac{4x^4}{1-x^4} + \frac{6x^6}{1-x^6}\right.$$

$$\left. - \frac{7x^7}{1+x^7} - \frac{8x^8}{1-x^8} + \frac{9x^9}{1+x^9} + \&c\right\}$$

$$25\phi(x)\phi^3(x^5) - \frac{\phi^5(x)}{\phi(x^5)}$$

$$= 24 + 40\left(\frac{x}{1+x} - \frac{3x^3}{1+x^3} - \frac{7x^7}{1+x^7} + \frac{9x^9}{1+x^9} \&c\right)$$

$$\frac{\psi^5(x)}{\psi(x^5)} - 25x^2\psi(x)\psi^3(x^5)$$

$$= 1 + 5\left(\frac{x}{1+x} - \frac{2x^2}{1+x^2} - \frac{3x^3}{1+x^3} + \frac{4x^4}{1+x^4} + \&c\right)$$

$$\frac{\dfrac{\phi^5(x)}{\phi(x^5)} + 4\dfrac{\psi^5(x)}{\psi(x^5)}}{\phi(x)\phi^3(x^5) + 4x^2\psi(x)\psi^3(x^5)} = 5\frac{\phi^2(x)}{\phi^2(x^5)}$$

values of m, n & x,

$$\theta = \frac{n}{(m+1)x+1-n-}\,\frac{1(1-n)(1+x)}{(m+2)x+3-n-}\,\frac{2(2-n)(1+x)}{(m+3)x+5-n-}\ \&c$$

30. $$1+\frac{x}{n+1}+\frac{x^2}{(n+1)(n+2)}+\&c$$

$$= \frac{e^x\lfloor n}{x^n} - \frac{n}{x+}\,\frac{1-n}{1+}\,\frac{1}{x+}\,\frac{2-n}{1+}\,\frac{2}{x+}\,\frac{3-n}{1+}\ \&c$$

$$= \frac{e^x\lfloor n}{x^n} - \frac{n}{x+1-n-}\,\frac{1(1-n)}{x+3-n-}\,\frac{2(2-n)}{x+5-n-}\,\frac{3(3-n)}{x+7-n-}\ \&c$$

Cor. $$\frac{1}{n}-\frac{x}{\lfloor 1}\cdot\frac{1}{n+1}+\frac{x^2}{\lfloor 2}\cdot\frac{1}{n+2}-\frac{x^3}{\lfloor 3}\cdot\frac{1}{n+3}+\&c$$

$$= \frac{\lfloor n-1}{x^n} - \frac{e^{-x}}{x+}\,\frac{1-n}{1+}\,\frac{1}{x+}\,\frac{2-n}{1+}\ \&c = \frac{\lfloor n-1}{x^n} - \frac{e^{-x}}{x+1-n-}\,\frac{1(1-n)}{x+3-n-}\ \&c.$$

Ex. 1. $x-\frac{x^3}{3\lfloor 1}+\frac{x^5}{5\lfloor 2}-\&c = \frac{1}{2}\sqrt{\pi}$ when $x=\infty$

2. $x-\frac{x^3}{3^2\lfloor 1}+\frac{x^5}{5^2\lfloor 2}-\&c = \sqrt{\frac{\pi}{4}}\left(\frac{C_0}{2}+\log_e 2x\right)$ when $x=\infty$

31. $$1+\frac{x}{1.3}+\frac{x^2}{1.3.5}+\frac{x^3}{1.3.5.7}+\frac{x^4}{1.3.5.7.9}+\&c$$

$$= \sqrt{\frac{\pi}{2x}e^x} - \frac{1}{x+}\,\frac{1}{1+}\,\frac{2}{x+}\,\frac{3}{1+}\,\frac{4}{x+}\ \&c$$

$$= \sqrt{\frac{\pi}{2x}e^x} - \frac{1}{x+1-}\,\frac{1.2}{x+5-}\,\frac{3.4}{x+9-}\,\frac{5.6}{x+13-}\ \&c$$

Cor. $$\frac{1}{1+}\,\frac{1}{1+}\,\frac{2}{1+}\,\frac{3}{1+}\,\frac{4}{1+}\ \&c = \sqrt{\frac{\pi e}{2}} - \left(1+\frac{1}{1.3}+\frac{1}{1.3.5}+\&c\right)$$

32. $$\frac{x}{1\lfloor 1}-\frac{x^2}{2\lfloor 2}+\frac{x^3}{3\lfloor 3}-\frac{x^4}{4\lfloor 4}+\&c = C_0+\log_e x+\frac{e^{-x}}{x+}\,\frac{1}{1+}\cdots\frac{2}{x+}\,\frac{3}{1+}\ \&c$$

$$= C_0+\log_e x + e^{-x}\phi(x)$$

$$\phi(x) = \frac{1}{x+1-}\,\frac{1^2}{x+3-}\,\frac{2^2}{x+5-}\ \&c$$

$\phi(x+1) = x\{\phi(x)\}^2$ very nearly when x is great

$\phi(2)-\{\phi(1)\}^2 = \frac{1}{170}$ nearly.

$\phi(x) = \frac{1}{x+1}+\frac{1}{x^3+5x^2+2x+\theta}$ where $\theta=0$ when $x=$ [illegible]

[illegible]

$$\frac{\psi(x^7)\psi(x^9) - \psi(-x^7)\psi(-x^9)}{\psi(x)\psi(x^{63}) - \psi(-x)\psi(-x^{63})} = x^6.$$

$$\frac{\psi(x^5)\psi(x^{11}) - \psi(-x^5)\psi(-x^{11})}{\psi(x)\psi(x^{55}) + \psi(-x)\psi(-x^{55})} = x^5$$

$$\frac{\psi(x^3)\psi(x^{13}) - \psi(-x^3)\psi(-x^{13})}{\psi(x)\psi(x^{39}) + \psi(-x)\psi(-x^{39})} = x^3$$

$$x\psi^5(x)\psi(x^3) - 9x^2\psi(x)\psi^5(x^3)$$
$$= \frac{x}{1-x^2} - \frac{2^2x^2}{1-x^4} + \frac{4^2x^4}{1-x^8} - \frac{5^2x^5}{1-x^{10}} + \&c$$

$$9\phi(x)\phi^5(x^3) - \phi^5(x)\phi(x^3)$$
$$= 8\left(1 + \frac{x}{1+x} - \frac{2^2x^2}{1-x^2} + \frac{4^2x^4}{1-x^4} - \frac{5^2x^5}{1+x^5} + \frac{7^2x^7}{1+x^7} - \&c\right)$$

If $n = \frac{1}{14-\theta}$ then $x = 100n - 8.6 + \frac{.81}{9n+.14}$ very nearly

If $x > 8$, $x = \frac{.08\theta^2 + 6.34\theta - 4.45}{14-\theta}$ to 2 places of decimals

and if $x < 7$, $14 - \theta = \frac{1000}{11x+75}$ to a place of decimal.

$x = 16.74$ when $\theta = 10$ and $\theta = 5.6$ when $x = 4$.

$$\phi(x) = \frac{1}{x} - \frac{\lfloor 1}{x^2} + \frac{\lfloor 2}{x^3} - \dots \pm \frac{1}{x^n} \cdot \frac{\lfloor n}{x+n+1 - \frac{1(1+n)}{x+n+3 - \frac{2(2+n)}{x+n+5 - \&c}}}$$

Cor. $\frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2}\left(1+\frac{1}{2}\right) + \frac{x^3}{\lfloor 3}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \&c = e^x(C_0 + \log_e x) + \phi(x)$

33. $\frac{x}{1^n \lfloor 1} - \frac{x^2}{2^n \lfloor 2} + \frac{x^3}{3^n \lfloor 3} - \frac{x^4}{4^n \lfloor 4} + \&c = \phi_n(x) + (-1)^{n-1}\psi_n(x)e^{-x}$

where $\phi_n(x)$ is the term independent of p in $\frac{x^p \lfloor -p}{p^n}$ and

$$\psi_n(x) - \psi_n'(x) = \frac{\psi_{n-1}(x)}{x}$$

$$\phi_n(x) = \frac{1}{\lfloor n}\left\{A_0(\log_e x)^n + nA_1(\log_e x)^{n-1} + \frac{n(n-1)}{\lfloor 2}A_2(\log_e x)^{n-2} + \dots + A_n\right\}$$

where $\lfloor x = A_0 - A_1\frac{x}{\lfloor 1} + A_2\frac{x^2}{\lfloor 2} - A_3\frac{x^3}{\lfloor 3} + \&c$

$$A_n = S_0 A_{n-1} + (n-1)S_2A_{n-2} + (n-1)(n-2)S_3A_{n-3} + \&c$$

$$\lfloor x = 1 - .5772156649x + .9890560173x^2 - .9074790803x^3 + .9817280965\frac{x^4}{1+\theta x}$$

$\theta = 1$ very nearly. when $x = 0$ $\theta = 1.00027$

$x = 1$; $\theta = \frac{51}{52}$. $x = 2$; $\theta = \frac{77}{82}$. $x = 6$; $\theta = \frac{5}{68}$. $x = 7$; $\theta =$ —

$$\psi_n(x) = \frac{x}{\left\{x + \frac{n}{2 + \frac{5n+10}{6x + \frac{41n+58}{10 + \&c}}}\right\}^{n+1}}$$

Ex. $\left(\frac{x}{1^2\lfloor 1} - \frac{x^2}{2^2\lfloor 2} + \frac{x^3}{3^2\lfloor 3} - \frac{x^4}{4^2\lfloor 4} + \frac{x^5}{5^2\lfloor 5} - \&c\right)$

$- \frac{1}{2}\left(\frac{x}{1\lfloor 1} - \frac{x^2}{2\lfloor 2} + \frac{x^3}{3\lfloor 3} - \frac{x^4}{4\lfloor 4} + \frac{x^5}{5\lfloor 5} - \dots\right)^2$

$= \frac{\pi^2}{12}$ when x becomes infinitely great

$$1 - \frac{n}{\lfloor 1}\left(\frac{1}{2}\right)^2 + \frac{n(n-1)}{\lfloor 2}\left(\frac{1.3}{2.4}\right)^2 - \&c$$

$$= \left(\frac{\lfloor n-\frac{1}{2}}{\lfloor -\frac{1}{2} \lfloor n}\right)^2 \left\{ 1 + \frac{1^2}{2.4} \cdot \frac{n.(n+\frac{1}{2})}{(n-\frac{1}{2})^2} + \right.$$

$$\left. \frac{1^2.5^2}{2.4.6.8} \cdot \frac{n(n-1).(n+\frac{1}{2})(n-\frac{1}{2})}{(n-\frac{1}{2})^2(n-\frac{5}{2})^2} + \&c \right\}$$

$$\frac{\pi}{4}\left\{1 + \left(\frac{1}{2}\right)^2\left(\frac{1+x}{2}\right) + \left(\frac{1.3}{2.4}\right)^2\left(\frac{1+x}{2}\right)^2 + \&c\right\}^2$$

$$- \frac{\pi}{4}\left\{1 + \left(\frac{1}{2}\right)^2\left(\frac{1-x}{2}\right) + \left(\frac{1.3}{2.4}\right)^2\left(\frac{1-x}{2}\right)^2 + \&c\right\}^2$$

$$= x + \frac{x^3}{2} + \frac{41x^5}{120} + \frac{21x^7}{80} + \&c$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{x \sin\theta \, d\theta \, d\phi}{(1-x^2\sin^2\theta)\sqrt{1+x^2\sin^2\theta\sin^2\phi}} = f(x)$$

$$\text{then } \phi(1-2x^4) = \frac{5}{(1+x)^2} f\left\{2\left(\frac{1-x}{1+x}\right)^2 - 1\right\}$$

$$- \frac{4}{(1+x)^2} f\left\{2\left(\frac{1-x}{1+x}\right)^2 - 1\right\}$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{x \sin\theta \, d\theta \, d\phi}{\sqrt{1-x^2\sin^2\theta}\sqrt{1-x^2\sin^2\theta\sin^2\phi}}$$

$$= \frac{2}{\pi} \cdot \int_0^{\frac{\pi}{2}}\int_0^{\sin^{-1}x} \frac{d\theta}{\sqrt{1-x^2\sin^2\phi - \sin^2\theta\cos^2\phi}} \, d\phi.$$

CHAPTER XV

1. $\left[1+\left(\frac{1}{2}\right)^2\left\{1-\frac{\phi(-x)}{\phi(x)}\right\}+\left(\frac{1.3}{2.4}\right)^2\left\{1-\frac{\phi(-x)}{\phi(x)}\right\}^2+\&c\right]\frac{1}{\sqrt{\phi(x)}}$ is always an even function of x

2. $1+\left(\frac{1}{2}\right)^2\left(1-\frac{1}{x}\right)+\left(\frac{1.3}{2.4}\right)^2\left(1-\frac{1}{x}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left(1-\frac{1}{x}\right)^3+\&c$
$=\sqrt{x}\left\{1+\left(\frac{1}{2}\right)^2(1-x)+\left(\frac{1.3}{2.4}\right)^2(1-x)^2+\left(\frac{1.3.5}{2.4.6}\right)^2(1-x)^3+\&c\right\}$

3. $1+\left(\frac{1}{2}\right)^2\left(\frac{4x}{1+2x+x^2}\right)+\left(\frac{1.3}{2.4}\right)^2\left(\frac{4x}{1+2x+x^2}\right)^2+\&c$
$=(1+x)\left\{1+\left(\frac{1}{2}\right)^2x^2+\left(\frac{1.3}{2.4}\right)^2x^4+\left(\frac{1.3.5}{2.4.6}\right)^2x^6+\&c\right\}$

Cor. $1+\left(\frac{1}{2}\right)^2\left\{\frac{8x(1+x^2)}{(1+x)^4}\right\}+\left(\frac{1.3}{2.4}\right)^2\left\{\frac{8x(1+x^2)}{(1+x)^4}\right\}^2+\&c$
$=(1+x)^2\left\{1+\left(\frac{1}{2}\right)^2x^4+\left(\frac{1.3}{2.4}\right)^2x^8+\left(\frac{1.3.5}{2.4.6}\right)^2x^{12}+\&c\right\}$

4. $1+\left(\frac{1}{2}\right)^2\frac{2x}{1+x}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{2x}{1+x}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left(\frac{2x}{1+x}\right)^3+\&c$
$=\sqrt{1+x}\left(1+\frac{1.3}{4^2}x^2+\frac{1.3.5.7}{4^2.8^2}x^4+\&c\right)$

5. $1+\left(\frac{1}{2}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)+\left(\frac{1.3}{2.4}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c$
$=1+\left(\frac{1}{4}\right)^2x+\left(\frac{1.5}{4.8}\right)^2x^2+\left(\frac{1.5.9}{4.8.12}\right)^2x^3+\&c$

6. $1+\left(\frac{1}{2}\right)^3x+\left(\frac{1.3}{2.4}\right)^3x^2+\left(\frac{1.3.5}{2.4.6}\right)^3x^3+\&c$
$=\left\{1+\left(\frac{1}{2}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)+\left(\frac{1.3}{2.4}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c\right\}^2$

7. Let $\pi\alpha\beta=1$ and $\alpha=\frac{\sqrt{\pi}}{\left(|\underline{-\frac{1}{4}}\right)^2}$ such that

$\alpha=1.180340,599016,092$ ✓

$\beta=.269676,300594,191$

$1/\beta=3.708149,354602,731$ ✓

$1+\left(\frac{1}{2}\right)^2\left(\frac{1+x}{2}\right)+\left(\frac{1.3}{2.4}\right)^2\left(\frac{1+x}{2}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left(\frac{1+x}{2}\right)^3+\&c$ ✓
$=\alpha\left\{1+\frac{1^2}{2.4}x^2+\frac{1^2.5^2}{2.4.6.8}x^4+\frac{1^2.5^2.9^2}{2.4.6.8.10.12}x^6+\&c\right\}$
$+\beta\left\{x+\frac{3^2}{4.6}x^3+\frac{3^2.7^2}{4.6.8.10}x^5+\frac{3^2.7^2.11^2}{4.6.8.10.12.14}x^7+\&c\right\}$

N.B. There are two solutions for this from &

$$\psi(x)\,\psi(x^5) - \psi(-x)\,\psi(-x^5)$$

$$= 2x\, f(x^2, x^4)\, f(x^{40}, x^{80}) + 4x^{15}\, \psi(x^6)\, \psi(x^{120})$$

$$\psi(p)\,\psi(q) = \psi(pq) + p\, f\left(\frac{q}{p}, p^2\right) + p^3 q\, f\left(\frac{q^2}{p^2}, \frac{p^3}{q}\right)$$

$$+ p^6 q^3\, f\left(\frac{q^3}{p^3}, \frac{p^4}{q^2}\right) + \&c$$

$$\phi(p)\,\phi(q) = \phi(pq) + 2p\, f\left(\frac{q}{p}, p^3 q\right) +$$

$$2pq\, f\left(\frac{p^3}{q}, \frac{q^3}{p}\right) + 2p^4 q\, f\left(\frac{q^3}{p^3}, \frac{p^5}{q}\right) +$$

$$2p^4 q^4\, f\left(\frac{p^5}{q^3}, \frac{q^5}{p^3}\right) + 2p^9 q^4\, f\left(\frac{q^5}{p^5}, \frac{p^7}{q^3}\right)$$

$$+ 2p^9 q^9\, f\left(\frac{p^7}{q^5}, \frac{q^7}{p^5}\right) + \&c$$

$$\sqrt{1+n^2}\left\{1 + \left(\frac{1}{2}\right)^2 \frac{1+in}{2} + \&c\right\}$$

$$= \frac{1+i}{2}\left\{1 + \left(\frac{1}{2}\right)^2 \frac{1 + \frac{n}{\sqrt{1+n^2}}}{2} + \&c\right\}$$

$$+ \frac{1-i}{2}\left\{1 + \left(\frac{1}{2}\right)^2 \frac{1 - \frac{n}{\sqrt{1+n^2}}}{2} + \&c\right\}.$$

8. $1+\left(\frac{1}{2}\right)^2\left\{\frac{(1+x)^2}{2(1+x^4)}\right\}+\left(\frac{1.3}{2.4}\right)^2\left\{\frac{(1+x)^2}{2(1+x^4)}\right\}^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left\{\frac{(1+x)^2}{2(1+x^4)}\right\}^3+\&c$

$$=\alpha\sqrt{1+x^2}\left(1+\frac{1}{2}\cdot\frac{1}{3}x^4+\frac{1.3}{2.4}\cdot\frac{1.5}{3.7}x^8+\frac{1.3.5}{2.4.6}\cdot\frac{1.5.9}{3.7.11}x^{12}+\&c\right)$$

$$+2\beta x\sqrt{1+x^2}\left(1+\frac{1}{2}\cdot\frac{3}{5}x^4+\frac{1.3}{2.4}\cdot\frac{3.7}{5.9}x^8+\frac{1.3.5}{2.4.6}\cdot\frac{3.7.11}{5.9.13}x^{12}+\&c\right)$$

Ex. 1. $1+\left(\frac{1}{2}\right)^2\frac{x}{1+x}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{x}{1+x}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left(\frac{x}{1+x}\right)^3+\&c$

$$=\sqrt{\frac{1+x}{1-x}}\left\{1-\frac{1.3}{4^2}\cdot\left(\frac{4x}{1-2x+x^2}\right)+\frac{1.3.5.7}{4^2 8^2}\cdot\left(\frac{4x}{1-2x+x^2}\right)^2-\&c\right\}$$

2. $1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\left(\frac{1.3.5}{2.4.6}\right)^2x^3+\&c$

$$=\frac{1}{\sqrt{1+x}}\left[1+\frac{1.3}{4^2}\left\{\frac{4x}{(1+x)^2}\right\}+\frac{1.3.5.7}{4^2 8^2}\left\{\frac{4x}{(1+x)^2}\right\}^2+\&c\right]$$

3. $1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\left(\frac{1.3.5}{2.4.6}\right)^2x^3+\&c$

$$=\frac{1}{\sqrt{1-x}}\left[1-\left(\frac{1}{4}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}+\left(\frac{1.5}{4.8}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}^2-\&c\right]$$

4. $1+\left(\frac{1}{2}\right)^2\frac{1-\sqrt{1-x}}{2}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c$

$$=\sqrt{1-x}\left\{1+\left(\frac{3}{4}\right)^2x+\left(\frac{3.7}{4.8}\right)^2x^2+\left(\frac{3.7.11}{4.8.12}\right)^2x^3+\&c\right\}$$

5. $1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\left(\frac{1.3.5}{2.4.6}\right)^2x^3+\&c$

$$=\frac{(1+x)}{(1-x)\sqrt{1-x}}\left[1-\left(\frac{3}{4}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}+\left(\frac{3.7}{4.8}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}^2-\&c\right]$$

6. $1+\left(\frac{1}{2}\right)^2\frac{1+x}{2}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{1+x}{2}\right)^2+\&c$

$$=\frac{\alpha}{\sqrt[4]{1-x^2}}\left\{1-\frac{1^2}{2.4}\cdot\frac{x^2}{1-x^2}+\frac{1^2.5^2}{2.4.6.8}\left(\frac{x^2}{1-x^2}\right)^2-\&c\right\}$$

$$+\frac{\beta}{\sqrt[4]{1-x^2}}\cdot\frac{x}{\sqrt{1-x^2}}\left\{1-\frac{3^2}{4.6}\cdot\frac{x^2}{1-x^2}+\frac{3^2.7^2}{4.6.8.10}\left(\frac{x^2}{1-x^2}\right)^2-\&c\right\}$$

7. $1+\left(\frac{1}{2}\right)^2\left\{\frac{(1+x)^2}{2(1+x^4)}\right\}+\left(\frac{1.3}{2.4}\right)^2\left\{\frac{(1+x)^2}{2(1+x^4)}\right\}^2+\&c$

$$=\frac{\alpha}{\sqrt{1-x^2}}\left\{1-\frac{1}{2}\cdot\frac{2}{3}\left(\frac{x^4}{1-x^4}\right)+\frac{1.3}{2.4}\cdot\frac{4.6}{3.7}\left(\frac{x^4}{1-x^4}\right)^2-\&c\right\}$$

$$+\frac{2\beta x}{\sqrt{1-x^2}}\left\{1-\frac{1}{2}\cdot\frac{2}{5}\left(\frac{x^4}{1-x^4}\right)+\ \ldots\right.$$

$x\psi^2(x)\psi^2(x^3)$

$$= \frac{x}{1-x^2} + \frac{2x^2}{1-x^4} + \frac{4x^4}{1-x^8} + \frac{5x^5}{1-x^{10}} + \&c$$

$\phi^2(x)\phi^2(x^3)$

$$= 1 + 4\left(\frac{x}{1-x} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \frac{7x^7}{1-x^7} + \frac{8x^8}{1-x^8} + \&c\right)$$

$x\psi(x^2)\psi(x^6)$

$$= \frac{x}{1-x^2} - \frac{x^5}{1-x^{10}} + \frac{x^7}{1-x^{14}} - \frac{x^{11}}{1-x^{22}} + \frac{x^{13}}{1-x^{26}} - \&c$$

$x\psi(x)\psi(x^7)$

$$= \frac{x}{1-x} - \frac{x^3}{1-x^3} - \frac{x^5}{1-x^5} + \frac{x^9}{1-x^9} + \frac{x^{11}}{1-x^{11}} - \frac{x^{13}}{1-x^{13}}$$

$$+ \frac{x^{15}}{1-x^{15}} - \frac{x^{17}}{1-x^{17}} - \frac{x^{19}}{1-x^{19}} + \frac{x^{23}}{1-x^{23}} \&c$$

$\psi(x^3)\psi(x^5) - \psi(-x^3)\psi(-x^5) = 2x^3\psi(x^2)\psi($[illegible]

$\psi(x)\psi(x^3) - \psi(-x)\psi(-x^3) = 2x\phi(x^2)\psi($[illegible]

$\psi(x)\psi(x^{11}) - \psi(-x)\psi(-x^{11})$

$$= 2xf(x^2, x^{10})f(x^{44}, x^{88}) + 2x^{15}\phi(x^6)\psi(x^{132}).$$

$$\phi(x)\phi(x^7) = 1 + 2\left(\frac{x}{1-x} - \frac{x^2}{1-x^2} - \frac{x^3}{1-x^3} + \frac{x^4}{1-x^4}\right.$$

$$\left. - \frac{x^5}{1-x^5} + \frac{x^6}{1-x^6} + \frac{x^8}{1-x^8} + \frac{x^9}{1-x^9} + \frac{x^{10}}{1-x^{10}}\right.$$

$1 + \left(\ldots\right) + \left(\frac{1.3}{\ldots}\right) \ldots$

$= 2(2-\sqrt{2})\left\{1 + \left(\frac{1}{2}\right)^2(\sqrt{2}-1)^4 + \left(\frac{1.3}{2.4}\right)^2(\sqrt{2}-1)^8 + \ldots\right.$

9. $1 + \left(\frac{1}{2}\right)^2\frac{80}{81} + \left(\frac{1.3}{2.4}\right)^2\left(\frac{80}{81}\right)^2 + \&c = \frac{18}{\sqrt{62}}\left\{1 + \frac{1.3}{4^2}\cdot\frac{1}{31^2} + \frac{1.3.5.7}{4^2.8^2}\cdot\frac{1}{31^4} \ldots\right.$

10. $1 + \left(\frac{1}{2}\right)^2\left(\frac{15}{16}\right) + \left(\frac{1.3}{2.4}\right)^2\left(\frac{15}{16}\right)^2 + \&c = \frac{32}{\sqrt{322}}\left\{1 + \frac{1.3}{4^2}\cdot\frac{1}{161^2} + \frac{1.3.5.7}{4^2.8^2}\cdot\frac{1}{161^4} \ldots\right.$

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11. $1 + \left(\frac{1}{2}\right)^2\frac{1}{3} + \left(\frac{1.3}{2.4}\right)^2\frac{1}{5} + \left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{7} + \&c$

$= \dfrac{1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \&c}{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c}$

12. $1 + \left(\frac{1}{2}\right)^2\frac{1}{5} + \left(\frac{1.3}{2.4}\right)^2\frac{1}{9} + \left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{13} + \&c = \dfrac{\lfloor -\frac{3}{4}}{\left(\lfloor -\frac{1}{4}\right)^3\sqrt{2}} = \alpha^2$

13. $1 - \left(\frac{1}{2}\right)^3 + \left(\frac{1.3}{2.4}\right)^3 - \left(\frac{1.3.5}{2.4.6}\right)^3 + \&c = \left(\dfrac{\lfloor \frac{1}{8}}{\lfloor \frac{1}{4}\,\lfloor -\frac{1}{8}}\right)^2$

14. $1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1.3}{2.4}\right)^3 - 13\left(\frac{1.3.5}{2.4.6}\right)^3 + \&c = \frac{2}{\pi}$

15. $1 + \left(\frac{1}{4}\right)^2 + \left(\frac{1.5}{4.8}\right)^2 + \left(\frac{1.5.9}{4.8.12}\right)^2 + \&c = \dfrac{\sqrt{\pi}}{\left(\lfloor -\frac{1}{4}\right)^2} = \alpha$

16. $1 - \left(\frac{1}{4}\right)^2 + \left(\frac{1.5}{4.8}\right)^2 - \left(\frac{1.5.9}{4.8.12}\right)^2 + \&c = \dfrac{\lfloor \frac{1}{8}}{[illegible]}$

17. $1 + \left(\frac{1}{4}\right)^3 + \left(\frac{1.5}{4.8}\right)^3 + \left(\frac{1.5.9}{4.8.12}\right)^3 + \&c = \dfrac{\sqrt{\pi}}{\lfloor \frac{1}{8}\left(\lfloor -\frac{1}{8}\right)^2\sqrt[4]{2}}$

18. $1 - 9\left(\frac{1}{4}\right)^3 + 17\left(\frac{1.5}{4.8}\right)^3 - 25\left(\frac{1.5.9}{4.8.12}\right)^3 + \&c = \frac{2\sqrt{2}}{\sqrt{\pi}}$

19. $1 + 9\left(\frac{1}{4}\right)^4 + 17\left(\frac{1.5}{4.8}\right)^4 + 25\left(\frac{1.5.9}{4.8.12}\right)^4 + \&c =$ [illegible]

20. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta\, d\phi}{\sqrt{1 - \sin^2\theta\,\sin^2\phi\,\sin^2\psi}} = \left\{\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \sin^2\theta\,\sin^2\ldots}}\right.$

21. $\displaystyle\int_0^{\frac{\pi}{4}}\left\{\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \sin^2\theta\,\sin^2\phi}}\right\}^2 d\phi = \frac{\pi^3}{16}\left\{1 + \left(\frac{1}{2}\right)^4 + \left(\frac{1.3}{2.4}\right)^4 + \&c\right.$

9. If $\phi(x)$ can be expressed in n different ways ... value in the nth way being $c_n + v_n$ and if $c_1, c_2, \ldots$ to be similar and $v_1, v_2, v_3 \ldots v_n$ are ... to be ... then $c_1, c_2, c_3 \ldots c_n$ must be identically equal ... and the real value of $\phi(x) = c + v_1$ ...

+ &c

$$\phi(n)\,\lfloor\!\underline{\;a+n}\,\lfloor\!\underline{\;b+n}\,\lfloor\!\underline{\;-n-1} - \frac{\phi(n+1)}{\lfloor\!\underline{\;1}}\,\lfloor\!\underline{\;a+n+1}\,\lfloor\!\underline{\;b+n+1}\,\lfloor\!\underline{\;-n-2} + \&c$$

$$\lfloor\!\underline{\;a+n}\,\lfloor\!\underline{\;b+n}\left\{\phi(0)\,\frac{\lfloor\!\underline{\;a}\,\lfloor\!\underline{\;b}}{\lfloor\!\underline{\;a+b+n+1}} + \frac{\phi(0)-\phi(1)}{\lfloor\!\underline{\;1}}\,\frac{\lfloor\!\underline{\;a+1}\,\lfloor\!\underline{\;b+1}}{\lfloor\!\underline{\;a+b+n+2}}\right.$$

$$\left. + \frac{\phi(0)-2\phi(1)+\phi(2)}{\lfloor\!\underline{\;2}}\cdot\frac{\lfloor\!\underline{\;a+2}\,\lfloor\!\underline{\;b+2}}{\lfloor\!\underline{\;a+b+n+2}} + \&c\right\}$$

The irreducible part in $1 - 24\left\{\frac{y}{(1-y)^2}\cos 2\theta\right.$ 2

$$\left. + \frac{y^2}{(1-y^2)^2}\cos 4\theta + \&c\right\} = \frac{\theta^2}{6}\left\{1 - 24\left(\frac{y}{1-y} + \frac{2y^2}{1-y^2} + \&c\right)\right\}$$

$$+ \theta\left\{1 - 24\left(\frac{y}{1-y} + \frac{2y^2}{1-y^2} + \&c\right)\right\}\left\{\cot\theta + 4\left(\frac{y}{1-y}\sin 2\theta\right.\right.$$

$$\left.\left. + \frac{y^2}{1-y^2}\sin 4\theta + \&c\right)\right\}.$$

and $Q_n = \phi(n) + \frac{n+1}{\lfloor\underline{1}}\phi(n+1) + \frac{[illegible]}{\lfloor\underline{2}}\phi(n+2) + \&c$

$\phi(0) + (1-x)\phi(1) + (1-x)^2\phi(2) + (1-x)^3\phi(3) + \&c$

$= Q_0 - Q_1 x + Q_2 x^2 - Q_3 x^3 + \&c$

$+ \frac{1}{\left(\log_e \frac{1}{1-x}\right)^{n+1}}\left\{P_0 + P_1 \log_e \frac{1}{1-x} + P_2\left(\log_e \frac{1}{1-x}\right)^2 + P_3\left(\log_e \frac{1}{1-x}\right)^3 + \&c\right\}$

Cor. 1. If $Q'_n = \frac{\lfloor\underline{m}}{\lfloor\underline{m-n}\,\lfloor\underline{n}}\phi(m) + \frac{\lfloor\underline{m+1}}{\lfloor\underline{m-n+1}\,\lfloor\underline{n}}\phi(m+1) + \frac{\lfloor\underline{m+2}}{\lfloor\underline{m-n+2}\,\lfloor\underline{n}}\phi(m+2) + \&c$

then $\phi(m)(1-x)^m + \phi(m+1)(1-x)^{m+1} + \phi(m+2)(1-x)^{m+2} + \&c$

$= Q_0 - Q_1 x + Q_2 x^2 - Q_3 x^3 + \&c$

$+ \frac{1}{\left(\log_e \frac{1}{1-x}\right)^{n+1}}\left\{P_0 + P_1 \log_e \frac{1}{1-x} + P_2\left(\log_e \frac{1}{1-x}\right)^2 + \&c\right\}$

Cor 2. If $a+b+c+1 = d+e$, then

$\left\{\frac{\lfloor\underline{a}\,\lfloor\underline{b}\,\lfloor\underline{c}}{\lfloor\underline{d}\,\lfloor\underline{e}} + \frac{1-x}{\lfloor\underline{1}}\cdot\frac{\lfloor\underline{a+1}\,\lfloor\underline{b+1}\,\lfloor\underline{c+1}}{\lfloor\underline{d+1}\,\lfloor\underline{e+1}} + \frac{(1-x)^2}{\lfloor\underline{2}}\,\frac{\lfloor\underline{a+2}\,\lfloor\underline{b+2}\,\lfloor\underline{c+2}}{\lfloor\underline{d+2}\,\lfloor\underline{e+2}} + \&c\right.$

$+ \log_e x$ when $x = 0$

$= -\Sigma\frac{1}{a} - \Sigma\frac{1}{b} + 1\cdot\frac{(c-d)(c-e)}{(a+1)(b+1)} + \frac{1}{2}\cdot\frac{(c-d)(c-d-1)(c-e)\ldots}{(a+1)(a+2)\ldots(b+1)\ldots} + \&c$

11. $\lfloor\underline{a}\,\lfloor\underline{b}\left\{\frac{\lfloor\underline{a+n}\,\lfloor\underline{b+n}}{\lfloor\underline{a+b+n+1}} + \frac{1-x}{\lfloor\underline{1}}\cdot\frac{\lfloor\underline{a+n+1}\,\lfloor\underline{b+n+1}}{\lfloor\underline{a+b+n+2}} + \&c\right\}.$

$= \left\{\lfloor\underline{a+n}\,\lfloor\underline{b+n}\,\lfloor\underline{-n-1} - \frac{x}{\lfloor\underline{1}}\lfloor\underline{a+n+1}\,\lfloor\underline{b+n+1}\,\lfloor\underline{-n-2} + \&c\right\}$

$+ \frac{1}{x^n}\left\{\lfloor\underline{a}\,\lfloor\underline{b}\,\lfloor\underline{n-1} - \frac{x}{\lfloor\underline{1}}\lfloor\underline{a+1}\,\lfloor\underline{b+1}\,\lfloor\underline{n-2} + \&c\right\}$

This is true for all values of n.

N.B. If n is an integer R.H.S takes the form $\infty - \infty$; should write $n+h$ for n first of all and then simplification h should be made to vanish.

$\ldots \delta(\delta+1) \ldots (1-x) \ldots + \&c$ Then

$$\frac{1}{\lfloor\gamma} + \frac{\alpha\beta}{\lfloor 1 \lfloor\gamma+1} x + \frac{\alpha(\alpha+1)(\beta)(\beta+1)}{\lfloor 2 \lfloor\gamma+2} x^2 + \&c$$

$$\frac{dy}{dx} = - \frac{\lfloor\delta-1 \lfloor\alpha-1 \lfloor\beta-1}{\lfloor\alpha-1 \lfloor\beta-1 \, x^{\gamma+1} (1-x)^{\delta} \left\{ \frac{1}{\lfloor\gamma} + \frac{\alpha\beta}{\lfloor 1 \lfloor\gamma+1} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\lfloor 2 \lfloor\gamma+2} x^2 \right.}$$

If $x^5 = (1+x)(1+x+x^4)$

then $x = \dfrac{e^{\frac{\pi}{24}\sqrt{47}}}{\sqrt{2}} (1+e^{-\pi\sqrt{47}})(1+e^{-3\pi\sqrt{47}})(1+e^{-5\pi\sqrt{47}})$ &c

Cor. 1. If n is a positive integer,

$$\lfloor\underline{a}\,\lfloor\underline{b}\left\{\frac{\lfloor\underline{a+n}\,\lfloor\underline{b+n}}{\lfloor\underline{a+b+n+1}}+\frac{1-x}{\lfloor\underline{1}}\frac{\lfloor\underline{a+n+1}\,\lfloor\underline{b+n+1}}{\lfloor\underline{a+b+n+2}}+\&c\right\}$$

$$+(-1)^n(\log_e x)\left\{\frac{\lfloor\underline{a+n}\,\lfloor\underline{b+n}}{[illegible]}+\frac{x}{\lfloor\underline{1}}\frac{\lfloor\underline{a+n+1}\,\lfloor\underline{b+n+1}}{\lfloor\underline{n+1}}+\&c\right\}$$

$$+(-1)^n\left\{\frac{\lfloor\underline{a+n}\,\lfloor\underline{b+n}}{[illegible]}\left(\Sigma\frac{1}{a+n}+\Sigma\frac{1}{b+n}-\Sigma\frac{1}{n}-0\right)\right.$$

$$+\frac{x}{\lfloor\underline{1}}\frac{\lfloor\underline{a+n+1}\,\lfloor\underline{b+n+1}}{\lfloor\underline{n+1}}\left(\Sigma\frac{1}{a+n+1}+\Sigma\frac{1}{b+n+1}-\Sigma\frac{1}{n+1}-\Sigma\frac{1}{1}\right)$$

$$+\frac{x^2}{\lfloor\underline{2}}\frac{\lfloor\underline{a+n+2}\,\lfloor\underline{b+n+2}}{\lfloor\underline{n+2}}\left(\Sigma\frac{1}{a+n+2}+\Sigma\frac{1}{b+n+2}-\Sigma\frac{1}{n+2}-\Sigma\frac{1}{2}\right)$$

$$\left.+\ \&c\right\}$$

$$=\frac{1}{x^n}\left\{\lfloor\underline{a}\,\lfloor\underline{b}\,\lfloor\underline{n-1}-\frac{x}{\lfloor\underline{1}}\lfloor\underline{a+1}\,\lfloor\underline{b+1}\,\lfloor\underline{n-2}+\&c\text{ to } n \text{ terms}\right\}$$

Cor 2. If n is a negative integer,

$$\lfloor\underline{a}\,\lfloor\underline{b}\left\{\frac{\lfloor\underline{a+n}\,\lfloor\underline{b+n}}{\lfloor\underline{a+b+n+1}}+\frac{1-x}{\lfloor\underline{1}}\frac{\lfloor\underline{a+n+1}\,\lfloor\underline{b+n+1}}{\lfloor\underline{a+b+n+2}}+\&c\right\}$$

$$+(-x)^{-n}(\log_e x)\left\{\frac{\lfloor\underline{a}\,\lfloor\underline{b}}{\lfloor\underline{-n}}+\frac{x}{\lfloor\underline{1}}\cdot\frac{\lfloor\underline{a+1}\,\lfloor\underline{b+1}}{\lfloor\underline{-n+1}}+\&c\right\}$$

$$+(-x)^{-n}\left\{\frac{\lfloor\underline{a}\,\lfloor\underline{b}}{\lfloor\underline{-n}}\left(\Sigma\frac{1}{a}+\Sigma\frac{1}{b}-\Sigma\frac{1}{-n}-0\right)\right.$$

$$+\frac{x}{\lfloor\underline{1}}\frac{\lfloor\underline{a+1}\,\lfloor\underline{b+1}}{\lfloor\underline{-n+1}}\left(\Sigma\frac{1}{a+1}+\Sigma\frac{1}{b+1}-\Sigma\frac{1}{-n+1}-\Sigma\frac{1}{1}\right)$$

$$+\frac{x^2}{\lfloor\underline{2}}\frac{\lfloor\underline{a+2}\,\lfloor\underline{b+2}}{\lfloor\underline{-n+2}}\left(\Sigma\frac{1}{a+2}+\Sigma\frac{1}{b+2}-\Sigma\frac{1}{-n+2}-\Sigma\frac{1}{2}\right)$$

$$\left.+\ \&c\right\}$$

$$=\lfloor\underline{a+n}\,\lfloor\underline{b+n}\,\lfloor\underline{-n-1}-\frac{x}{\lfloor\underline{1}}\lfloor\underline{a+n+1}\,\lfloor\underline{b+n+1}\,\lfloor\underline{-n-2}+\&c\ [illegible]$$

$$\lfloor a \lfloor b \left\{ \phi(0) \frac{\lfloor a \lfloor b}{\lfloor a+b+1} + \frac{\phi(0)-\phi(1)}{\lfloor 1} \cdot \frac{\lfloor a+1 \lfloor b+1}{\lfloor a+b+3} + \&c \right\}$$

$$+ \phi'(0) \lfloor a \lfloor b + \phi'(1) \frac{\lfloor a+1 \lfloor b+1}{\lfloor 1 \lfloor 1} + \phi'(2) \frac{\lfloor a+2 \lfloor b+2}{\lfloor 2 \lfloor 2} + \&c$$

$$+ \phi(0) \lfloor a \lfloor b \left(\Sigma \frac{1}{a} + \Sigma \frac{1}{b} - 2 \Sigma \frac{1}{1} \right) + \phi(1) \frac{\lfloor a+1 \lfloor b+1}{\lfloor 1 \lfloor 1} \times$$

$$\left(\Sigma \frac{1}{a+1} + \Sigma \frac{1}{b+1} - 2 \Sigma \frac{1}{1} \right) + \phi(2) \frac{\lfloor a+2 \lfloor b+2}{\lfloor 2 \lfloor 2} \left(\Sigma \frac{1}{a+2} + \Sigma \frac{1}{b+2} - 2 \Sigma \frac{1}{2} \right) + \&c$$

$$= 0.$$

If $$y = \frac{\lfloor \alpha-1 \lfloor \beta-1}{\lfloor \alpha+\beta-1} \cdot \frac{1 + \frac{\alpha}{\lfloor 1} \cdot \frac{\beta}{\alpha+\beta}(1-x) + \frac{\alpha(\alpha+1)}{\lfloor 2} \frac{\beta(\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)}(1-x)^2 + \&c}{1 + \frac{\alpha}{\lfloor 1} \cdot \frac{\beta}{\lfloor 1} x + \frac{\alpha(\alpha+1)}{\lfloor 2} \frac{\beta(\beta+1)}{\lfloor 2} x^2 + \&c}$$

then $$\frac{dy}{dx} = -\frac{1}{x(1-x)^{\alpha+\beta} \left\{ 1 + \frac{\alpha}{\lfloor 1} \cdot \frac{\beta}{\lfloor 1} x + \frac{\alpha(\alpha+1)}{\lfloor 2} \frac{\beta(\beta+1)}{\lfloor 2} x^2 + \&c \right\}^2}$$

~~If $\alpha+\beta = \gamma$, then~~ If $\alpha+\beta+1 = \gamma+\delta$.

$$\int \frac{\int x^{n-2} \left\{ 1 + \frac{\alpha}{\lfloor 1} \cdot \frac{\beta}{\gamma} x + \frac{\alpha(\alpha+1)}{\lfloor 2} \cdot \frac{\beta(\beta+1)}{\gamma(\gamma+1)} x^2 + \&c \right\} dx}{x^\gamma (1-x)^\delta \left\{ 1 + \frac{\alpha}{\lfloor 1} \cdot \frac{\beta}{\gamma} x + \frac{\alpha(\alpha+1)}{\lfloor 2} \cdot \frac{\beta(\beta+1)}{\gamma(\gamma+1)} x^2 + \&c \right\}^2} dx$$

$$= \frac{x^{n-\gamma}(1-x)^{1-\delta}}{(n-\gamma)(n-1)} \left\{ 1 + \frac{(n-\alpha)(n-\beta)}{n(n-\gamma+1)} x + \frac{(n-\alpha)(n-\alpha+1)(n-\beta)(n-\beta+1)}{n(n+1)(n-\gamma+1)(n-\gamma+2)} x^2 + \&c \right\}$$

$$\div \left\{ 1 + \frac{\alpha}{\lfloor 1} \cdot \frac{\beta}{\gamma} x + \frac{\alpha(\alpha+1)}{\lfloor 2} \cdot \frac{\beta(\beta+1)}{\gamma(\gamma+1)} x^2 + \quad \right\}$$

$$2.\ \lfloor a \lfloor b \left\{ \frac{\lfloor a+1}{\lfloor a+b+1} + \frac{[illegible]}{\lfloor 1} [illegible] \frac{[illegible]}{\lfloor 2} [illegible] \lfloor a+b+3 + \&c \right\}$$

$$+\left(\log_e x\right)\left\{ \lfloor a \lfloor b + x\,\frac{\lfloor a+1 \lfloor b+1}{\lfloor 1 \lfloor 1} + x^2\,\frac{\lfloor a+2 \lfloor b+2}{\lfloor 2 \lfloor 2} + \&c \right\}$$

$$+ \lfloor a \lfloor b \left(\Sigma \frac{1}{a} + \Sigma \frac{1}{b} - 2\Sigma 0 \right) + x\,\frac{\lfloor a+1 \lfloor b+1}{\lfloor 1 \lfloor 1}\left(\Sigma \frac{1}{a+1} + \Sigma \frac{1}{b+1} - 2\Sigma 1 \right)$$

$$+ x^2\,\frac{\lfloor a+2 \lfloor b+2}{\lfloor 2 \lfloor 2}\left(\Sigma \frac{1}{a+2} + \Sigma \frac{1}{b+2} - 2\Sigma \frac{1}{2} \right) + \&c$$

$$= 0.$$

Cor. $\pi \left\{ 1 + \left(\frac{1}{2}\right)^2 (1-x) + \left(\frac{1.3}{2.4}\right)^2 (1-x)^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 (1-x)^3 + \&c \right\}$

$$= \left(\log_e \frac{16}{x}\right)\left\{ 1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 x^3 + \&c \right\}$$

$$- 4\left\{ \left(\frac{1}{2}\right)^2 \frac{1}{1.2}\, x + \left(\frac{1.3}{2.4}\right)^2 \left(\frac{1}{1.2} + \frac{1}{3.4} \right) x^2 + \&c \right\}$$

Ex. 1. $4 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\tan \frac{\phi}{2}}{\sqrt{1 - k \cos^2\theta \cdot \cos^2\phi}}\, d\theta\, d\phi$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-(1-k)\sin^2\theta}} + \log_e k \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k\sin^2\theta}}$$

2. $\left\{ \frac{\lfloor a \lfloor b}{\lfloor a+b+1} + \frac{\lfloor a+1 \lfloor b+1}{\lfloor 1 \lfloor a+b+2} + \frac{\lfloor a+2 \lfloor b+2}{\lfloor 2 \lfloor a+b+3} + \&c \text{ to } n \text{ terms} \right\}$

$- \left\{ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c \text{ to } n \text{ terms} \right\}$ when $n = \infty$

$$= - \Sigma \frac{1}{a} - \Sigma \frac{1}{b}$$

3. $\pi \left\{ 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 + \&c \right\} - \left(1 + \frac{1}{2} + \frac{1}{3} + \&c\right) = $ [illegible]

13. If $\phi(n) = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 + \&c$ to n terms, then

i. $\pi\, \phi\left(\frac{n}{4} + \frac{1}{4}\right) = 3\log_e 2 + \Sigma \frac{1}{\frac{n-1}{2}} + \frac{3}{4n^2} - \frac{79}{32n^4} + \frac{997}{32n^6} - \frac{61223}{[illegible]}$

$$= 5\log_e 2 + 2\Sigma \frac{1}{\frac{n-1}{2}} - \Sigma \frac{1}{\frac{n-1}{2}} - \frac{2}{8n^2+13} \text{ [illegible]}$$

ii. $\phi(n) = \left(\frac{\lfloor n-\frac{1}{2}}{\lfloor n-1} \right)^2 \left\{ \frac{1}{n} + \left(\frac{1}{2}\right)^2 \frac{1}{n+1} + \left(\frac{1.3}{2.4}\right)^2 [illegible] \right.$

If $\alpha+\beta+1=\gamma+\delta$

and $\phi(x) = \dfrac{1+\dfrac{\alpha}{\underline{|1}}\cdot\dfrac{\beta}{\delta}(1-x)+\dfrac{\alpha(\alpha+1)}{\underline{|2}}\cdot\dfrac{\beta(\beta+1)}{\delta(\delta+1)}(1-x)^2+\&c}{1+\dfrac{\alpha}{\underline{|1}}\cdot\dfrac{\beta}{\gamma}x+\dfrac{\alpha(\alpha+1)}{\underline{|2}}\cdot\dfrac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2+\&c}$, then

$$\phi'(x) = -\frac{\underline{|\gamma-1}\;\underline{|\delta-1}}{\underline{|\alpha-1}\;\underline{|\beta-1}}\cdot\frac{1}{x^{\gamma}(1-x)^{\delta}\left\{1+\dfrac{\alpha}{\underline{|1}}\cdot\dfrac{\beta}{\gamma}x+\dfrac{\alpha(\alpha+1)}{\underline{|2}}\cdot\dfrac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2+\&c\right\}^2}$$

$$\left\{1+\frac{\alpha}{\underline{|1}}\cdot\frac{\beta}{\gamma}\left(\frac{1-\sqrt{1-x}}{2}\right)+\frac{\alpha(\alpha+1)}{\underline{|2}}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}\cdot\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c\right\}$$

$$\times\left\{1+\frac{\alpha}{\underline{|1}}\cdot\frac{\beta}{\delta}\left(\frac{1-\sqrt{1-x}}{2}\right)+\frac{\alpha(\alpha+1)}{\underline{|2}}\cdot\frac{\beta(\beta+1)}{\delta(\delta+1)}\cdot\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c\right\}$$

$$=1+\frac{\alpha}{\gamma}\cdot\frac{\beta}{\delta}\cdot\frac{\alpha+\beta+1}{2}\cdot\frac{\alpha+\beta}{2\alpha+2\beta}x+\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\cdot\frac{\beta(\beta+1)}{\delta(\delta+1)}\cdot\frac{(\alpha+\beta+1)(\alpha+\beta+3)}{2\cdot 4}\cdot\frac{(\alpha+\beta)(\alpha+\beta+\ldots}{(2\alpha+2\beta)(2\alpha+2\ldots}$$

$$\times x^2+\&c$$

iii. $\phi(\frac{1}{2}) = \frac{1}{2}$.

iv. $\phi(\frac{1}{2}) = \dfrac{\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c}{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c}$

v. $\phi(n+1) = \phi(n) + \frac{1}{\pi}\left(\frac{\lfloor n-\frac{1}{2}}{\lfloor n}\right)^2$

vi. $1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2.4}{3.5}\right)^2 + \left(\frac{2.4.6}{3.5.7}\right)^2 + \&c$ to n terms $= \frac{\pi^2}{4}\phi(n+\frac{1}{2})$

$- 2\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c\right)$

vii. $1 + (\frac{1}{3})^2\left\{1 + \left(\frac{3}{5}\right)^2 + \left(\frac{3.7}{5.9}\right)^2 + \left(\frac{3.7.11}{5.9.13}\right)^2 + \&c \text{ to } n \text{ terms}\right\} = 2\phi(n+$ [illegible]

14. $\frac{a}{1-m} + \frac{(a+b)m}{1-mx} + \frac{(a+2b)m^2}{1-mx^2} + \frac{(a+3b)m^3}{1-mx^3} + \&c$

$= a.\frac{1-mn}{(1-m)(1-n)} + (a+b)\frac{1-mnx^2}{(1-mx)(1-nx)}(mnx) + (a+2b)\frac{1-mnx^4}{(1-mx^2)(1-nx^2)}$

$\times(mnx^2)^2 + (a+3b)\frac{1-mnx^6}{(1-mx^3)(1-nx^3)}(mnx^3)^3 + \&c$

$+ \frac{b}{m}\left\{\frac{mn}{(1-n)^2} + \frac{(mnx)^2}{(1-nx)^2} + \frac{(mnx^2)^3}{(1-nx^2)^2} + \frac{(mnx^3)^4}{(1-nx^3)^2} + \&c\right\}$

Cor. 1. $\frac{a}{1-m} + \frac{(a+b)m}{1-mx} + \frac{(a+2b)m^2}{1-mx^2} + \&c$

$= a.\frac{1+m}{1-m} + (a+b)\frac{1+mx}{1-mx}(m^2x) + (a+2b)\frac{1+mx^2}{1-mx^2}(m^2x^2)^2 + \&c$

$+ b\left\{\frac{m}{(1-m)^2} + \frac{m^3x^2}{(1-mx)^2} + \frac{m^5x^6}{(1-mx^2)^2} + \frac{m^7x^{12}}{(1-mx^3)^2} + \&c\right\}$

Cor 2. If A_n denotes the no. of factors in n including 1 and n

then $\frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \frac{A_4}{x^4} + \&c$

$= \frac{1}{x-1} + \frac{1}{x^2-1} + \frac{1}{x^3-1} + \frac{1}{x^4-1} + \&c$

hence show that $\frac{1}{x-1} + \frac{1}{x^2-1} + \frac{1}{x^3-1} + \&c$

$= \frac{1}{x}.\frac{x+1}{x-1} + \frac{1}{x^4}.\frac{x^2+1}{x^2-1} + \frac{1}{x^9}.\frac{x^3+1}{x^3-1} + \&c$

Cor 3. $\frac{1}{e^{2x}-1} + \frac{1}{e^{4x}-1} + \frac{1}{e^{6x}-1} + \&c = e^{-x}\coth x + e^{-2x}\coth 2x + e^{-3x}$ [illegible]

$+ \&c$

15. $\frac{1+b}{1-a}.\frac{1+bx}{1-ax}.\frac{1+bx^2}{1-ax^2}.\frac{1+bx^3}{1-ax^3}.\&c$

$= 1 + \frac{a+b}{1-x} + \frac{a+b}{1-x}.\frac{a+bx}{1-x^2} + \frac{a+b}{1-x}.\frac{a+bx}{1-x^2}.\frac{a+bx^2}{1-x^3}$ [illegible]

$$f(x^3, x^6) = \psi(x) - x\psi(x^9)$$

$$(a+1)(b+1)(c+1) + (a-1)(b-1)(c-1)$$
$$= 2(a+b+c+abc)$$

$$\frac{\phi^2(x)}{\phi^2(-x)} + \frac{\phi^2(y)}{\phi^2(-y)} + \frac{\phi^2(z)}{\phi^2(-z)} + \frac{\phi^2(x)\phi^2(y)\phi^2(z)}{\phi^2(-x)\phi^2(-y)\phi^2(-z)}$$
$$= 4\frac{\phi^2(x^4)\phi^2(y^4)\phi^2(z^4)}{\phi^2(-x)\phi^2(-y)\phi^2(-z)} + 256xyz\frac{\psi^2(x^4)\psi^2(y^4)\psi^2(z^4)}{\phi^2(-x)\phi^2(-y)\phi^2(-z)}$$

$$f(-x^2, -x^3)f(-x, -x^4) = f(-x, -x^2)f(-x^5, -x^{10})$$

$$f(-x, -x^6)f(-x^2, -x^5)f(-x^3, -x^4)$$
$$= f(x, -x^2)f^2(-x^7, -x^{14})$$

$$f(-x, -x^5) = \psi(x^3)\sqrt[3]{\frac{\phi(-x)}{\psi(x)}}$$

16. Let $f(p, q) = 1 + (p+q) + pq(p^2+q^2) + (pq)^3(p^3+q^3) + (pq)^6(p^4+q^4)$
$+ (pq)^{10}(p^5+q^5) + \&c$, then

i. $f(p,q) = f(q,p)$. ii. $f(1,p) = 2f(p,p^3)$ iii. $f(-1,p) = 0$.

iv. If n is any integer, then

$$f(p,q) = p^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}} f\left\{p(pq)^n, q(pq)^{-n}\right\}$$

These 4 results are evident from the series itself.

v. $f(p,q) = (1+p)(1+kp)(1+k^2p)(1+k^3p)\ \&c$
$\times (1+q)(1+kq)(1+k^2q)(1+k^3q)\ \&c$
$\times (1-pq)(1-kpq)(1-k^2pq)(1-k^3pq)\ \&c$
where $k = pq$.

Sol. Since $f(-1,p) = 0$ by iii, we see from iv that if $p(pq)^n$ or $q(pq)^n = -1$, then $f(p,q) = 0$ $\therefore (1+pk^n)\ \&\ (1+qk^n)$ are factors of $f(p,q)$. Again we see that if $(pq)^n = 1$, then
$f(p,q)\left\{1 - \left(\frac{p}{q}\right)^{\frac{n}{2}}\right\} = 0$ $\therefore f(p,q) = 0$ $\therefore 1 - (pq)^n$ is a factor. $f(p,q) = \prod_{n=1}^{n=\infty}(1+pk^n)(1+qk^n)(1-pqk^n)$

vi. $f(p, k^{n-1}q) \times f(kp, k^{n-2}q) \times f(k^2p, k^{n-3}q) \times \cdots f(k^{n-1}p, q)$
$$= f(p,q)\frac{\{f(-k^n, -k^{2n})\}^n}{f(-k, -k^2)}$$

vii. $f(p,q) - f\left(p^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}}, p^{\frac{n(n-1)}{2}} q^{\frac{n(n+1)}{2}}\right)$
$= pf\left\{\frac{q}{p}, p^{\frac{n(n+3)}{2}} q^{\frac{n(n+1)}{2}}\right\} + p^3q f\left\{\frac{q^2}{p^2}, p^{\frac{n(n+5)}{2}} q^{\frac{n(n+3)}{2}}\right\}$
$+ p^6q^3 f\left\{\frac{q^3}{p^3}, p^{\frac{n(n+7)}{2}} q^{\frac{n(n+5)}{2}}\right\} + \&c$ to $(n-1)$ terms.

viii. If $pq = rs$, $f(p,q)f(r,s) + f(-p,-q)f(-r,-s)$
$= 2f(pr, qs)f(qr, ps)$

ix. If $pq = rs$, $f(p,q)f(r,s) - f(-p,-q)f(-r,-s)$
$= 2pf\left\{\frac{s}{p}, \frac{p}{s}(pq)\ldots\right\} f\left\{\frac{s}{q}, \frac{q}{s}(pq)\ldots\right\}$

$$e^{-\frac{\pi}{\sqrt{2}} \cdot \frac{1+\frac{1.3}{4^2}(1-x^2)+\frac{1.3.5.7}{4^2.8^2}(1-x^2)^2+\&c}{1+\frac{1.3}{4^2}x^2+\frac{1.3.5.7}{4^2.4^2}x^4+\&c}}$$

$$= F\left(\frac{2x}{1+x}\right).$$

$$1 - \frac{f^5(-x,-x^2)}{f(-x^5,-x^{10})} = 5x\, d \log_e \frac{f(-x^2,-x^3)}{f(-x,-x^4)} \Big/ dx$$

$$f(u_1,v_1) = f(u_n,v_n) + u_1 f\left(\frac{v_{n-1}}{u_1}, \frac{u_{n+1}}{u_1}\right) + v_1 f\left(\frac{u_{n-1}}{v_1}, \frac{v_{n+1}}{v_1}\right)$$

$$+ u_2 f\left(\frac{v_{n-2}}{u_2}, \frac{u_{n+2}}{u_2}\right) + v_2 f\left(\frac{u_{n-2}}{v_2}, \frac{v_{n+2}}{v_2}\right) + \&c$$

If $ab = cd$, then

$$f(a,b) f(c,d) f\left(an, \frac{b}{n}\right) f\left(cn, \frac{d}{n}\right)$$

$$- f(-a,-b) f(-c,-d) f\left(-an, -\frac{b}{n}\right) f\left(-cn, -\frac{d}{n}\right)$$

$$= 2a f\left(\frac{c}{a}, ad\right) f\left(\frac{d}{an}, acn\right) f\left(n, \frac{ab}{n}\right) \psi(ab)$$

(Deduction).

x. $f(p, kq)\, f(q, kp) = f(p, q)\, f(k, k^3)$ where $k = pq$

xi. $f(p, q) + f(-p, -q) = 2 f(p^3 q, p q^3)$

xii. $f(p, q) - f(-p, -q) = 2p\, f\left(\frac{q}{p}, \frac{p}{q} k^4\right)$ where $k = pq$

xiii. $f(p, q)\, f(-p, -q) = f(-p^2, -q^2)\, f(-pq, -pq)$

xiv. $f^2(p, q) + f^2(-p, -q) = 2 f(p^2, q^2)\, f(pq, pq)$

xv. $f^2(p, q) - f^2(-p, -q) = 4p\, f\left(\frac{q}{p}, \frac{p}{q} k^4\right) f(k^2, k^6)$ where $k = pq$

N.B. Of all the functions the forms of $f(p, q)$, the most important are $f(-x, -x)$, $f(-x, -x^2)$ and $f(-x, -x^3)$

$f(x, x) = \phi(x)$ and $f(x, x^3) = \psi(x)$.

Ex. 1. $f(-x, x) = \phi(-x^4)$

2. $f(x, x^7)\, f(x^3, x^5) = \psi(x)\, \psi(x^4)$

3. $2f(x^3, x^5) = \psi(\sqrt{x}) + \psi(-\sqrt{x})$

4. $2f(x, x^7) = \dfrac{\psi(\sqrt{x}) - \psi(-\sqrt{x})}{\sqrt{x}}$

17. i $\phi(x) = 1 + 2x + 2x^4 + 2x^9 + 2x^{16} + \&c$

$$= \frac{1+x}{1-x} \cdot \frac{1-x^2}{1+x^2} \cdot \frac{1+x^3}{1-x^3} \cdot \frac{1-x^4}{1+x^4} \&c$$

ii. $\psi(x) = 1 + x + x^3 + x^6 + x^{10} + \&c$

$$= \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^3} \cdot \frac{1-x^6}{1-x^5} \&c$$

iii $\frac{1}{2} \log_e \phi(x) = \dfrac{x}{1+x} + \dfrac{x^3}{3(1+x^3)} + \dfrac{x^5}{5(1+x^5)} + \&c$

iv $\log_e \psi(x) = \dfrac{x}{1+x} + \dfrac{x^2}{2(1+x^4)} + \dfrac{x^3}{3(1+x^3)} + \&c$

v. $\dfrac{\psi(x)}{\phi(x)} = \dfrac{1+x^2}{1+x} \cdot \dfrac{1+x^4}{1+x^3} \cdot \dfrac{1+x^6}{1+x^5} \&c$

$$\phi(x)+\phi(x^2) = \frac{1 \quad 2}{(1-x)(1-x^4)(1-x^7)(1-x^9)(1-x^{12})(1-x^{15})(1-x^{17})^2}$$

$$\phi(x)-\phi(x^2) =$$

If $F(x^2) = t$

then $F\left(\frac{2x}{1+x}\right) = \frac{\sqrt{t}}{1+(1-t)(1-t+\frac{3}{4}t^2)}$ nearly.

$$\left(x+\frac{x^2}{2}+\frac{21x^3}{64}+\frac{31x^4}{128}+\&\right)^{11}$$

$$= x^{11}+\frac{11}{2}x^{12}+\frac{1111}{64}x^{13}+\frac{11111}{896}+\frac{111111}{2688}x^{14}+\&$$

θ is less than $\frac{1}{2161}\left\{\frac{\log x}{8+\cdot 14(\log x)^2}\right\}^5$

or $F(1-e^{-x}) = \frac{x}{10+\sqrt{36+x^2}} - \frac{1}{2160}\cdot\left(\frac{x}{8+\cdot 14x^2}\right)^5$

$$\frac{1}{\phi(x^2)} = \frac{1}{\phi(x)\pm\phi(x^2)} + \frac{1}{\phi(-x)\pm\phi(x^2)}$$

$$\pm\frac{1}{\phi(-x^2)} = \frac{1}{\phi(x)\pm\phi(-x^2)} + \frac{1}{\phi(-x)\pm\phi(-x^2)}$$

vi $\phi(x) + \phi(-x) = 2\phi(x^4)$

vii $\phi(x) - \phi(-x) = 4x\,\psi(x^8)$

viii $\phi(x)\,\phi(-x) = \phi^2(-x^2)$

ix $\phi(x)\,\psi(x^2) = \psi^2(x)$

x. $\phi^2(x) - \phi^2(-x) = 8x\,\psi^2(x^4)$

xi. $\phi^2(x) + \phi^2(-x) = 2\,\phi^2(x^2)$

xii. $\phi^4(x) - \phi^4(-x) = 16x\,\psi^4(x^2)$.

xiii. $\psi^2(x) + \psi^2(-x) = 2\,\psi(x^2)\,\phi(x^4)$

xiv. If $\left(\frac{1-z}{1+z}\right)^2 = \left\{\frac{\phi(-x)}{\phi(x)}\right\}^4$ then $1 - z^2 = \left\{\frac{\phi(-x^2)}{\phi(x^2)}\right\}^4$

Ex. 1. $\frac{\psi(x)}{\psi(-x)} = \sqrt{\frac{\phi(x)}{\phi(-x)}}$.

2. $\psi(x)\,\psi(-x) = \psi(x^2)\,\phi(-x^2)$

3. $\frac{\psi(x)\,\psi(-x)}{\psi(x^4)\,\psi(-x^2)} = \frac{\psi(-x^2)}{\psi(x^4)}$

18. If $F(x) = e^{-\pi \cdot \frac{1 + \left(\frac{1}{2}\right)^2(1-x) + \left(\frac{1.3}{2.4}\right)^2(1-x)^2 + \left(\frac{1.3.5}{2.4.6}\right)^2(1-x)^3 + \&c}{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 x^3 + \&c}}$

i. $F(x) = \frac{x}{16}\, e^{4 \cdot \frac{\left(\frac{1}{2}\right)^2 \frac{1}{1.2} x + \left(\frac{1.3}{2.4}\right)^2\left(\frac{1}{1.2} + \frac{1}{3.4}\right)x^2}{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \&c}}$

ii. $F\left(1 - \frac{4}{x}\right) + \theta = \frac{\log_e x}{10 + \sqrt{36 + (\log_e x)^2}}$ ~~where θ is numerically much less than $\frac{2}{135}F^5\left(1-\frac{4}{x}\right)$~~ $\theta = \frac{1}{2160}\left\{\frac{\log_e x}{8 + \sqrt{(\log_e x)^2}}\right\}^5$.

iii. $\log_e F(x)\,\log_e F(1-x) = \pi^2$

iv. $F(1-x) + F\left(1 - \frac{1}{x}\right) = 0$

v. $F\left\{\frac{4x}{(1+x)^2}\right\} = \sqrt{F(x^2)}$

N. B. If we know the expansion of $F\left(\frac{2x}{1+x}\right)$ to n terms, then we can find it's expansion to 2n terms as follows. Suppose we want the expansion [illegible]

$$2\frac{\psi'(x)}{\psi(x)} - 2x\frac{\psi'(x^2)}{\psi(x^2)} = \frac{\phi'(x)}{\phi(x)}$$

$$\frac{\phi'(-x)}{\phi(-x)} - 4x\frac{\phi'(-x^2)}{\phi(-x^2)} = \frac{\phi'(x)}{\phi(x)}.$$

3/ $F(x) = z$ & $y = \sqrt{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \&c}$, then

$$\phi(z) = y; \quad \phi(-z) = y\sqrt[4]{1-x}.$$

$$\phi(z^2) = y\sqrt{\frac{1+\sqrt{1-x}}{2}}; \quad \phi(-z^2) = y\sqrt[8]{1-x}.$$

$$\phi(z^4) = y \cdot \frac{1+\sqrt[4]{1-x}}{2}.$$

$$\phi(\sqrt{z}) = y\sqrt{1+\sqrt{x}}; \quad \phi(-\sqrt{z}) = y\sqrt{1-\sqrt{x}}.$$

$$\phi(\sqrt[4]{z}) = y(1+\sqrt[4]{x}); \quad \phi(-\sqrt[4]{z}) = y(1-\sqrt[4]{x}).$$

$$\psi(z) = \frac{y}{\sqrt{2}}\sqrt[8]{\frac{x}{z}}.$$

$$\psi(z^2) = \frac{y}{2}\sqrt[4]{\frac{x}{z}}.$$

$$\psi(z^4) = \frac{y}{2}\sqrt{\frac{1-\sqrt{1-x}}{2z}}.$$

$$\psi(z^8) = y\cdot\frac{1-\sqrt[4]{1-x}}{4z}.$$

$$\psi(\sqrt{z}) = y\sqrt[4]{\frac{1+\sqrt{x}}{2}}\cdot\sqrt[16]{\frac{x}{z}}.$$

$$\psi(\sqrt[4]{z}) = y\sqrt{1+\sqrt[4]{x}}\cdot\sqrt[8]{\frac{1+\sqrt{x}}{2}}\cdot\sqrt[32]{\frac{x}{z}}.$$

Writing $\frac{x^2}{2-x^2}$ for x we have the expansion of $\Phi(x^2)$ to $2n$ terms i.e. that of $\left[\Phi\left\{\frac{4x}{(1+x)^2}\right\}\right]^2$ to $2n$ terms. Extracting the square root and expanding the result in ascending powers of $\frac{2x}{1+x^2}$ we can find the expansion of $\Phi\left(\frac{2x}{1+x^2}\right)$ to $2n$ terms.

vi. $$8\Phi\left(\frac{2x}{1+x}\right) = x + \frac{5}{16}x^3 + \frac{369}{2048}x^5 + \frac{4097}{32768}x^7 + \frac{1594895}{16777216}x^9 + \&c$$

vii. $$2F(1-e^{-8x}) = x - \frac{x^3}{3} + \frac{31}{120}x^5 - \frac{661}{2520}x^7 + \frac{219677}{725760}x^9 - \&c$$

viii. $$2F\left(1-e^{-\frac{8x}{1-x^2}}\right) = x + \frac{2}{3}x^3 + \frac{31}{120}x^5 + \frac{37}{1260}x^7 + \frac{5981}{725760}x^9 + \&c$$

Ex. 1 $\Phi(0)=0$, $F(\frac{1}{2}) = e^{-\pi}$, $F(1)=1$, $F\{(\sqrt{2}-1)^2\} = e^{-\pi\sqrt{2}}$

IX $$\phi^2(x) = 1 + \left(\frac{1}{2}\right)^2\left\{1 - \frac{\phi^4(-x)}{\phi^4(x)}\right\} + \left(\frac{1\cdot3}{2\cdot4}\right)^2\left\{1 - \frac{\phi^4(-x)}{\phi^4(x)}\right\}^2 + \&c$$

x. $$1 + \left(\frac{1}{2}\right)^2\left\{\frac{\phi^4(-x)}{\phi^4(x)}\right\} + \left(\frac{1\cdot3}{2\cdot4}\right)^2\left\{\frac{\phi^4(-x)}{\phi^4(x)}\right\}^2 + \&c$$
$$= \frac{\phi^2(x)}{n\phi^2(x^n)}\left\{1 + \left(\frac{1}{2}\right)^2\left[\frac{\phi^4(-x^n)}{\phi^4(x^n)}\right] + \left(\frac{1\cdot3}{2\cdot4}\right)^2\left[\frac{\phi^4(-x^n)}{\phi^4(x^n)}\right]^2 + \&c\right\}$$

xi. $$F\left\{\frac{\phi^4(-x^n)}{\phi^4(x^n)}\right\} = \sqrt[n]{F\left\{\frac{\phi^4(-x)}{\phi^4(x)}\right\}}$$

xii. $$F\left\{1 - \frac{\phi^4(-x)}{\phi^4(x)}\right\} = \sqrt[n]{F\left\{1 - \frac{\phi^4(-x^n)}{\phi^4(x^n)}\right\}}$$

xiii. $$F\left\{1 - \frac{\phi^4(-x)}{\phi^4(x)}\right\} = x$$

xiv. $$\phi^2\{F(x)\} = 1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot3}{2\cdot4}\right)^2 x^2 + \ldots$$

$$\begin{cases} 1+2e^{-\pi}+2e^{-4\pi}+2e^{-9\pi}+\&c = \dfrac{\sqrt[4]{\pi}}{\Gamma\left(\frac{3}{4}\right)} = k \\ 1-2e^{-\pi}+2e^{-4\pi}-2e^{-9\pi}+\&c = \dfrac{k}{\sqrt[4]{2}} \end{cases}$$

$$\begin{cases} 1+2(e^{-\pi})^2+2(e^{-4\pi})^2+2(e^{-9\pi})^2+\&c = \dfrac{k}{2}\sqrt{2+\sqrt{2}} \\ 1-2(e^{-\pi})^2+2(e^{-4\pi})^2-2(e^{-9\pi})^2+\&c = \dfrac{k}{\sqrt[8]{2}} \end{cases}$$

$$\begin{cases} 1+2(e^{-\pi})^4+2(e^{-4\pi})^4+2(e^{-9\pi})^4+\&c = \dfrac{k}{2}\left(1+\dfrac{1}{\sqrt[4]{2}}\right) \\ 1-2(e^{-\pi})^4+2(e^{-4\pi})^4-2(e^{-9\pi})^4+\&c = \dfrac{k}{\sqrt{2}}\sqrt[4]{\sqrt[4]{2}+\sqrt[4]{8}} \end{cases}$$

$$\begin{cases} 1+2\sqrt{e^{-\pi}}+2\sqrt{e^{-4\pi}}+2\sqrt{e^{-9\pi}}+\&c = \dfrac{k}{\sqrt[4]{2}}\sqrt{1+\sqrt{2}} \\ 1-2\sqrt{e^{-\pi}}+2\sqrt{e^{-4\pi}}-2\sqrt{e^{-9\pi}}+\&c = \dfrac{k}{\sqrt[4]{2}}\sqrt{\sqrt{2}-1} \end{cases}$$

$$\begin{cases} 1+2\sqrt[4]{e^{-\pi}}+2\sqrt[4]{e^{-4\pi}}+2\sqrt[4]{e^{-9\pi}}+\&c = k\left(1+\dfrac{1}{\sqrt[4]{2}}\right) \\ 1-2\sqrt[4]{e^{-\pi}}+2\sqrt[4]{e^{-4\pi}}-2\sqrt[4]{e^{-9\pi}}+\&c = k\left(1-\dfrac{1}{\sqrt[4]{2}}\right) \end{cases}$$

$$\begin{cases} 1+e^{-\pi}+e^{-3\pi}+e^{-6\pi}+e^{-10\pi}+\&c = k\sqrt[8]{\dfrac{e^{\pi}}{32}} \\ 1+(e^{-\pi})^2+(e^{-3\pi})^2+(e^{-6\pi})^2+(e^{-10\pi})^2+\&c = k\sqrt[4]{\dfrac{e^{\pi}}{32}} \end{cases}$$

$$\begin{cases} 1+(e^{-\pi})^4+(e^{-3\pi})^4+(e^{-6\pi})^4+(e^{-10\pi})^4+\&c = \dfrac{k}{4}\sqrt{e^{\pi}(2-\sqrt{2})} \\ 1+(e^{-\pi})^8+(e^{-3\pi})^8+(e^{-6\pi})^8+\&c = \dfrac{k}{4}e^{\pi}\left(1-\dfrac{1}{\sqrt[4]{2}}\right) \end{cases}$$

$$\begin{cases} 1+\sqrt{e^{-\pi}}+\sqrt{e^{-3\pi}}+\sqrt{e^{-6\pi}}+\&c = k\sqrt[4]{\dfrac{1+\sqrt[4]{2}}{2\sqrt{2}}}\sqrt[4]{e^{\pi}} \\ 1+\sqrt[4]{e^{-\pi}}+\sqrt[4]{e^{-3\pi}}+\sqrt[4]{e^{-6\pi}}+\&c = k\sqrt[32]{\dfrac{e^{\pi}}{2}}\sqrt[8]{1+\sqrt{2}}\sqrt{\dfrac{1+\sqrt[4]{2}}{\sqrt{8}}} \end{cases}$$

19. i.e. If $y = \pi \frac{1 + \left(\frac{1}{2}\right)^2 (1-x) + \left(\frac{1.3}{2.4}\right)^2 (1-x)^2 + \&c}{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \&c}$, then

$$1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 x^3 + \&c$$

$$= \left(1 + 2e^{-y} + 2e^{-4y} + 2e^{-9y} + 2e^{-16y} + \&c\right)^2$$

Cor. If $\sqrt{\alpha\beta} = \sqrt{\pi}$, then

$$\sqrt{\alpha}\left\{\frac{1}{2} + e^{-\alpha^2} + e^{-4\alpha^2} + e^{-9\alpha^2} + \&c\right\}$$

$$= \sqrt{\beta}\left\{\frac{1}{2} + e^{-\beta^2} + e^{-4\beta^2} + e^{-9\beta^2} + \&c\right\}$$

20. If $\alpha\beta = \pi$, then

$$\sqrt{\alpha}\left\{1 + \frac{2}{(1+\alpha^2)^{n+1}} + \frac{2}{(1+4\alpha^2)^{n+1}} + \frac{2}{(1+9\alpha^2)^{n+1}} + \&c\right\}$$

$$= \frac{\lfloor n-\frac{1}{2}}{\lfloor n}\sqrt{\beta}\left\{1 + 2e^{-2\beta}\phi(2\beta) + 2e^{-4\beta}\phi(4\beta) + \&c\right\}.$$

where $\phi(x) = 1 + \frac{n}{n}\cdot\frac{x}{\lfloor 1} + \frac{n(n-1)}{n(n-\frac{1}{2})}\frac{x^2}{\lfloor 2} + \frac{n(n-1)(n-2)}{n(n-\frac{1}{2})(n-1)}\frac{x^3}{\lfloor 3} + \&c$

to $n+1$ terms.

$$= \frac{1}{\lfloor 2n}\left\{(2x)^n \lfloor n + \frac{n}{\lfloor 1}(2x)^{n-1}\lfloor n+1 + \frac{n(n-1)}{\lfloor 2}(2x)^{n-2}\lfloor n+2 \right.$$
$$\left. + \&c \text{ to infinity}\right\}$$

Cor. If $\alpha\beta = \pi$, then

$$\sqrt{\alpha}\left\{\frac{1}{2} + e^{-\alpha^2} + e^{-4\alpha^2} + e^{-9\alpha^2} + \&c\right\}$$

$$= \sqrt{\beta}\left\{\frac{1}{2} + e^{-\beta^2} + e^{-4\beta^2} + e^{-9\beta^2} + \&c\right\}$$

Ex. 1. $\left(1 + 2e^{-\pi} + 2e^{-4\pi} + 2e^{-9\pi} + \&c\right)^2$

$$= 1 + \left(\frac{1}{2}\right)^2\frac{1}{2} + \left(\frac{1.3}{2.4}\right)^2\frac{1}{2^2} + \left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{2^3} + \&c = \frac{\sqrt{\pi}}{\left(\lfloor -\frac{1}{4}\right)^2}$$

2. $\frac{1}{2} + e^{-\pi\sqrt{2}} + e^{-4\pi\sqrt{2}} + e^{-9\pi\sqrt{2}} + \&c = \frac{\lfloor \frac{1}{8}}{\sqrt{\pi}\,\lfloor \frac{1}{4}}$

3. $\frac{\pi - \frac{1}{4}}{e^{\pi}} + \frac{4\pi - \frac{1}{4}}{e^{4\pi}} + \frac{9\pi - \frac{1}{4}}{e^{9\pi}} + \&c = \frac{1}{8}$

$$\begin{cases} (1-e^{-\pi})(1-e^{-2\pi})(1-e^{-3\pi})(1-e^{-4\pi})\ \&c = k\,\dfrac{\sqrt[24]{e^{\pi}}}{\sqrt[8]{8}} \\ (1-e^{-2\pi})(1-e^{-4\pi})(1-e^{-6\pi})(1-e^{-8\pi})\ \&c = \dfrac{k}{\sqrt[4]{2}}\sqrt[12]{e^{\pi}}. \\ (1-e^{-4\pi})(1-e^{-8\pi})(1-e^{-12\pi})\ \&c = \dfrac{k}{2}\sqrt[8]{2}\,\sqrt[6]{e^{\pi}}. \\ (1-e^{-8\pi})(1-e^{-16\pi})(1-e^{-24\pi})\ \&c = \dfrac{k}{2}\sqrt[3]{e^{\pi}}\,\sqrt[4]{\dfrac{1}{2}\left(\sqrt[4]{2}-\dfrac{1}{\sqrt[4]{2}}\right)}. \end{cases}$$

$$\begin{cases} (1-e^{-\pi})(1-e^{-3\pi})(1-e^{-5\pi})\ \&c = \dfrac{\sqrt[8]{2}}{\sqrt[24]{e^{\pi}}} \\ (1-e^{-2\pi})(1-e^{-6\pi})(1-e^{-10\pi})\ \&c = \dfrac{\sqrt[8]{8}}{\sqrt[12]{e^{\pi}}}. \\ (1-e^{-4\pi})(1-e^{-12\pi})(1-e^{-20\pi})\ \&c = \dfrac{\sqrt[4]{2(2+\sqrt{2})\sqrt[4]{2}}}{\sqrt[6]{e^{\pi}}}. \end{cases}$$

$$\begin{cases} (1+e^{-\pi})(1+e^{-3\pi})(1+e^{-5\pi})\ \&c = \dfrac{\sqrt[4]{2}}{\sqrt[24]{e^{\pi}}}. \\ (1+e^{-2\pi})(1+e^{-6\pi})(1+e^{-10\pi})\ \&c = \dfrac{\sqrt[4]{(1+\sqrt{2})\sqrt[4]{2}}}{\sqrt[12]{e^{\pi}}}. \end{cases}$$

$$\left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2\left(1+\frac{1}{3}\right)x^2 + \left(\frac{1.3.5}{2.4.6}\right)^2\left(1+\frac{1}{3}+\frac{1}{5}\right)x^3 + \&c$$

$$= -\frac{1}{4}\left\{1+\left(\frac{1}{2}\right)^2 x + \&c\right\}\log(1-x).$$

Reconsider

$$\left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2\left(1+\frac{1}{2}\right)x^2 + \&c$$

$$= -2\left\{1+\left(\frac{1}{2}\right)^2 x + \&c\right\}\left\{\log\left[1+\left(\frac{1}{2}\right)^2 x + \&c\right]\right.$$

$$\left. + \frac{1}{3}\log(1-x) - \frac{1}{12}\log x - \frac{\pi}{12}\left[1+\left(\frac{1}{2}\right)^2(1-x)+\&c\right]\right\}$$

$$= 4\left(\frac{1}{e^{2y}} + \frac{1}{2}\cdot\frac{1}{e^{2\cdot 2y}} + \&c\right)$$

CHAPTER XVI

1. $h\phi(h) + h\phi(2h) + h\phi(3h) + h\phi(4h) + \&c$

$$= \int_0^\infty \phi(x)\,dx + F(h) \quad \text{when } F(0)=0$$

F(h) can be found by expanding the left but writing the constant instead of a series.

Cor. If $h\phi(h) = ah^p + bh^q + ch^r + dh^s + \&c$ and if p, q, r, s &c be not negative, then

$$h\phi(h) + h\phi(2h) + h\phi(3h) + h\phi(4h) + \&c$$

$$= \int_0^\infty \phi(x)\,dx + a\frac{B_p}{p}h^p\cos\frac{\pi p}{2} + b\frac{B_q}{q}h^q\cos\frac{\pi q}{2} + \&c$$

N.B. F(h) is not a terminating series; but if p, q, r, s &c be odd integers F(h) appear to be 0 or to have some finite value. In this case F(h) is really a function of $e^{-\frac{\pi}{h}}$ which is 0 when $h=0$ but which can't be expressed as a function in ascending powers of h. When h is small the difference between the real and the apparent value of F(h) is so small that we can neglect it. e.g.

The apparent value of $x + \frac{2x}{1+x^2} + \frac{2x}{1+(2x)^2} + \&c$ $= \pi$, the real value being $\pi\left(1 + 2e^{-\frac{2\pi}{x}} + 2e^{-\frac{4\pi}{x}} + \&c\right)$

Ex. Shew that $x\left\{\phi(0) + [\phi(x)+\phi(-x)] + [\phi(2x)+\phi(-2x)] + \&c\right\}$

$$= \int_{-\infty}^{\infty}\phi(x)\,dx + F\left(e^{-\frac{\pi}{x}}\right) \quad \text{where } F(0)=0.$$

2. $\frac{1^{m-1}}{e^{1^n x}} + \frac{2^{m-1}}{e^{2^n x}} + \frac{3^{m-1}}{e^{3^n x}} + \frac{4^{m-1}}{e^{4^n x}} + \&c$

$$= \frac{1}{m}\frac{\underline{\left|\frac{m}{n}\right.}}{x^{\frac{m}{n}}} + \frac{B_m}{m}\cos\frac{\pi m}{2} - \frac{x}{\underline{|1}}\cdot\frac{B_{m+n}}{m+n}\cos\frac{\pi(m+n)}{2}$$

$$+ \frac{x^2}{\underline{|2}}\frac{B_{m+2n}}{m+2n}\cos\frac{\pi(m+2n)}{2} - \&c$$

$$\frac{1^m . 1^n}{e^x-1} + \frac{2^m(2^n+1^n)}{e^{2x}-1} + \frac{3^m(3^n+1^n)}{e^{3x}-1} + \frac{4^m(4^n+2^n+1^n)}{e^{4x}-1} + \&c$$

$$= \frac{\lfloor m}{x^{m+1}} S_{m+1} S_{m-n+1} + \frac{\lfloor n}{x^{n+1}} S_{n+1} S_{n-m+1}$$

$$+ \frac{1}{x} S_{1-m} S_{1-n} - \frac{1}{2} S_{-m} S_{-n} + \frac{B_2}{\lfloor 2} x S_{-1-m} S_{-1-n}$$

$$- \frac{B_4}{\lfloor 4} x^3 S_{-3-m} S_{-3-n} + \&c$$

$$1^m \{ 1^n e^{-x} + 2^n e^{-2x} + 3^n e^{-3x} + \&c \}$$
$$+ 2^m \{ 1^n e^{-2x} + 2^n e^{-4x} + 3^n e^{-6x} + \&c \}$$
$$+ 3^m \{ 1^n e^{-3x} + 2^n e^{-6x} + 3^n e^{-9x} + \&c \}$$
$$+ \quad \&c \quad \&c \quad \&c \quad \&c \quad \&c$$

$$= \frac{\lfloor m}{x^{m+1}} S_{1+m-n} + \frac{\lfloor n}{x^{n+1}} S_{1+n-m}$$

$$+ S_{-m} S_{-n} - \frac{x}{\lfloor 1} S_{-m-1} S_{-n-1} + \frac{x^2}{\lfloor 2} S_{-m-2} S_{-n-2} \&c$$

N.B. This is true only when n is positive.

$$3.\quad \frac{1^{l-1}}{(1+x^n)^m}+\frac{2^{l-1}}{(1+2^n x^n)^m}+\frac{3^{l-1}}{(1+3^n x^n)^m}+\&c$$

$$=\frac{\lfloor\underline{\tfrac{l}{n}}\;\lfloor\underline{m-\tfrac{l}{n}-1}}{l x^l\,\lfloor\underline{m-1}}+\frac{B_l}{l}\cos\frac{\pi l}{2}-\frac{m}{\lfloor\underline{1}}.x^n\frac{B_{l+n}}{l+n}\cos\frac{\pi(l+n)}{2}$$

$$+\frac{m(m+1)}{\lfloor\underline{2}}.x^{2n}\frac{B_{l+2n}}{l+2n}\cos\frac{\pi(l+2n)}{2}-\&c$$

$$4.\quad \frac{1^{m-1}}{e^{1^n x}-1}+\frac{2^{m-1}}{e^{2^n x}-1}+\frac{3^{m-1}}{e^{3^n x}-1}+\&c$$

$$=\frac{1}{m}.\frac{\lfloor\underline{\tfrac{m}{n}}}{x^{\frac{m}{n}}}S_{\frac{m}{n}}+\frac{1}{x}S_{1+n-m}-\frac{1}{2}\frac{B_m}{m}\cos\frac{\pi m}{2}$$

$$+\frac{x}{\lfloor\underline{1}}.\frac{B_2}{2}.\frac{B_{m+n}}{m+n}\cos\frac{\pi(m+n)}{2}-\frac{x^3}{\lfloor\underline{3}}.\frac{B_4}{4}.\frac{B_{m+3n}}{m+3n}\cos\frac{\pi(m+3n)}{2}$$

$$+\frac{x^5}{\lfloor\underline{5}}.\frac{B_6}{6}.\frac{B_{m+5n}}{m+5n}\cos\frac{\pi(m+5n)}{2}-\&c$$

$$\text{Cor.}\quad \frac{1^{n-1}}{e^{1^n x}-1}+\frac{2^{n-1}}{e^{2^n x}-1}+\frac{3^{n-1}}{e^{3^n x}-1}+\&c$$

$$=\frac{C_0-\frac{1}{n}\log_e x}{x}-\frac{1}{2}.\frac{B_n}{n}\cos\frac{\pi n}{2}+\frac{x}{\lfloor\underline{1}}\frac{B_2}{2}.\frac{B_{2n}}{2n}\cos\frac{2\pi n}{2}$$

$$-\frac{x^3}{\lfloor\underline{3}}.\frac{B_4}{4}.\frac{B_{4n}}{4n}\cos 2\pi n+\frac{x^5}{\lfloor\underline{5}}.\frac{B_6}{6}.\frac{B_{6n}}{6n}\cos 3\pi n-\&c$$

$$5.\quad \text{If } \phi(t)=\frac{1^{m-1}}{(e^{1^p x})^{t^q}}+\frac{2^{m-1}}{(e^{2^p x})^{t^q}}+\frac{3^{m-1}}{(e^{3^p x})^{t^q}}+\&c,\ \text{then}$$

$$1^{n-1}\phi(1)+2^{n-1}\phi(2)+3^{n-1}\phi(3)+4^{n-1}\phi(4)+\&c$$

$$=\left\{\frac{\lfloor\underline{\tfrac{m}{p}}}{m\,x^{\frac{m}{p}}}S_{1+\frac{m}{p}q-n}\right\}+\left\{\frac{\lfloor\underline{\tfrac{n}{q}}}{n\,x^{\frac{n}{q}}}S_{1+\frac{n}{q}p-m}\right\}$$

$$+\frac{B_m}{m}.\frac{B_n}{n}.\cos\frac{\pi m}{2}.\cos\frac{\pi n}{2}-\frac{x}{\lfloor\underline{1}}.\frac{B_{m+p}}{m+p}.\frac{B_{n+q}}{n+q}\cos\frac{\pi(m+p)}{2}\cos\frac{\pi(n+q)}{2}$$

$$+\frac{x^2}{\lfloor\underline{2}}.\frac{B_{m+2p}}{m+2p}\frac{B_{n+2q}}{n+2q}\cos\frac{\pi(m+2p)}{2}\cos\frac{\pi(n+2q)}{2}-\&c$$

$$\psi(x) - x\psi(x^9) = \frac{\phi(-x^9)}{\sqrt[3]{\phi(-x^3)}}\sqrt[3]{\psi(x^3)}.$$

$$\left\{3\phi(-x^9) - \phi(-x)\right\}^3 = 8\,\frac{\psi^3(x)}{\psi(x^3)}\,\phi(-x^3)$$

$$\frac{3617 + 16320\left(\frac{1^{15}x}{1-x} + \frac{2^{15}x^2}{1-x^2} + \frac{3^{15}x^3}{1-x^3} + \&c\right)}{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \&c\right)}$$

$$= 1617\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \&c\right)\right\}^3$$

$$+ 2000\left\{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \&c\right)\right\}^2$$

$$\frac{43867 - 28728\left(\frac{1^{17}x}{1-x} + \frac{2^{17}x^2}{1-x^2} + \frac{3^{17}x^3}{1-x^3} + \right)}{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \quad\right)}$$

$$= 38367\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \right)\right\}^3$$

$$+ 5500\left\{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \right)\right\}^2$$

$$\frac{174611 + 13200\left(\frac{1^{19}x}{1-x} + \frac{2^{19}x^2}{1-x^2} + \quad\right)}{1 + 480\left(\frac{1^7x}{1-x} + \frac{2^7x^2}{1-x^2} + \right)}$$

$$= 53361\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \right)\right\}^3$$

$$+ 121250\left\{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \right)\right\}^2$$

Cor. If $\frac{m}{p} = \frac{n}{q} = k$, R.H.S becomes

$$\frac{\lfloor k}{mnx^{k}}\left\{k\left(\Sigma\frac{1}{k-1} - C_0 - \log_e x\right) + C_0(m+n)\right\} + \frac{\lfloor 3m}{m}\cdot\frac{\lfloor 3n}{n}\cos\frac{\pi m}{2}\cos\frac{\pi n}{2} - \text{\&c}$$

6. $(1+2r\cos\theta+r^2)(1+2r^3\cos\theta+r^6)(1+2r^5\cos\theta+r^{10})$ &c
$\times(1-r^2)(1-r^4)(1-r^6)(1-r^8)$ &c
$= 1+2r\cos\theta+2r^4\cos 2\theta+2r^9\cos 3\theta+2r^{16}\cos 4\theta+$ &c

Cor. 1. $(1-r^3)(1-r^9)(1-r^{15})(1-r^{21})$ &c

$$= \frac{1+r-(r^4+2r^9+r^{16})+(r^{25}+2r^{36}+r^{49})-\text{\&c}}{1+r+r^3+r^6+r^{10}+r^{15}+r^{21}+\text{\&c}}$$

$$= \frac{1+r\omega-(r^4\omega+2r^9+r^{16}\omega)+(r^{25}\omega+2r^{36}+r^{49}\omega)-\text{\&c}}{1+r\omega+r^3+r^6+r^{10}\omega+r^{15}+r^{21}+r^{28}\omega+\text{\&c}}$$

$$= \frac{1-r^3-r^{15}+r^{24}+r^{48}-\text{\&c}}{1+r^9+r^{27}+r^{54}+\text{\&c}}$$

$$= \frac{1-2r^9+2r^{36}-2r^{81}+\text{\&c}}{1+r^3+r^6+r^{15}+r^{21}+\text{\&c}}$$

2. If $(1+r\sqrt{x}+r^2)(1+r^3\sqrt{x}+r^6)(1+r^5\sqrt{x}+r^{10})$ &c
$\times(1-r^2)(1-r^4)(1-r^6)(1-r^8)(1-r^{10})$ &c $= \phi(x)$

Find $\phi(1)$, $\phi(2)$, $\phi(3)$ and $\phi(4)$ in ascending powers of r

7. If $\alpha\beta = \pi^2$, then
$\sqrt[4]{\alpha}\, e^{-\frac{\alpha}{3}}(1-e^{-2\alpha})(1-e^{-4\alpha})(1-e^{-6\alpha})$ &c
$= \sqrt[4]{\beta}\, e^{-\frac{\beta}{3}}(1-e^{-2\beta})(1-e^{-4\beta})(1-e^{-6\beta})$ &c

8. If $\alpha\beta = \pi^2$
$\frac{\alpha}{12} - \frac{1}{4}\log_e\alpha + \frac{1}{1(e^{2\alpha}-1)} + \frac{1}{2(e^{4\alpha}-1)} + \frac{1}{3(e^{6\alpha}-1)} +$ &c
$= \frac{\beta}{12} - \frac{1}{4}\log_e\beta + \frac{1}{1(e^{2\beta}-1)} + \frac{1}{2(e^{4\beta}-1)} + \frac{1}{3(e^{6\beta}-1)} +$ &c

$$1 + 480\left(\frac{1^7 x}{1-x} + \frac{2^7 x^2}{1-x^2} + \frac{3^7 x^3}{1-x^3} + \&c\right)$$

$$= \left\{1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} + \&c\right)\right\}^2.$$

$$\left\{1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} + \&c\right)\right\}$$

$$\times \left\{1 - 504\left(\frac{1^5 x}{1-x} + \frac{2^5 x^2}{1-x^2} + \frac{3^5 x^3}{1-x^3} + \&c\right)\right\}$$

$$= 1 - 264\left(\frac{1^9 x}{1-x} + \frac{2^9 x^2}{1-x^2} + \frac{3^9 x^3}{1-x^3} + \&c\right).$$

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$$\left\{1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} + \&c\right)\right\}$$

$$\times \left\{1 - 264\left(\frac{1^9 x}{1-x} + \frac{2^9 x^2}{1-x^2} + \frac{3^9 x^3}{1-x^3} + \&c\right)\right\}$$

$$= 1 - 24\left(\frac{1^{13} x}{1-x} + \frac{2^{13} x^2}{1-x^2} + \frac{3^{13} x^3}{1-x^3} + \&c\right)$$

$$691 + 240.273\left(\frac{1^{11} x}{1-x} + \frac{2^{11} x^2}{1-x^2} + \frac{3^{11} x^3}{1-x^3} + \&c\right)$$

$$= 441\left\{1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} + \&c\right)\right\}^3$$

$$+ 250\left\{1 - 504\left(\frac{1^5 x}{1-x} + \frac{2^5 x^2}{1-x^2} + \frac{3^5 x^3}{1-x^3} + \&c\right)\right\}^2.$$

If $S_{2n} = \frac{B_{2n}}{4n}\cos\pi n + \frac{1^{2n-1} x}{1-x} + \frac{2^{2n-1} x^2}{1-x^2} + \&c$ then it is possible to find S_{2n} in the series containing $S_4 S_{2n-4}$, $S_6 S_{2n-6}$ &c and not containing $S_2 S_{2n-2}$.

9. If $\alpha\beta = \pi^2$

$$\frac{1}{2} + \frac{2\alpha}{e^{2\alpha}-1} + \frac{4\alpha}{e^{4\alpha}-1} + \frac{6\alpha}{e^{6\alpha}-1} + \frac{8\alpha}{e^{8\alpha}-1} + \&c$$

$$+ \frac{2\beta}{e^{2\beta}-1} + \frac{4\beta}{e^{4\beta}-1} + \frac{6\beta}{e^{6\beta}-1} + \frac{8\beta}{e^{8\beta}-1} + \&c$$

$$= \frac{\alpha+\beta}{12}.$$

Cor. $\frac{2\pi}{e^{2\pi}-1} + \frac{4\pi}{e^{4\pi}-1} + \frac{6\pi}{e^{6\pi}-1} + \&c = \frac{\pi}{12} - \frac{1}{4}.$

10. If $\alpha\beta = \pi$

$$e^{\frac{n^2}{4}}\sqrt{\alpha}\left\{\frac{1}{2} + e^{-\alpha^2}\cos n\alpha + e^{-4\alpha^2}\cos 2n\alpha + e^{-9\alpha^2}\cos 3n\alpha + \&c\right\}$$

$$= \sqrt{\beta}\left\{\frac{1}{2} + e^{-\beta^2}\cosh n\beta + e^{-4\beta^2}\cosh 2n\beta + e^{-9\beta^2}\cosh 3n\beta + \&c\right\}$$

11. If $\alpha\beta = \pi$

$$e^{n^2}\cdot\frac{\sin n\alpha\,(1-2e^{-2\alpha^2}\cos 2n\alpha + e^{-4\alpha^2})(1-2e^{-4\alpha^2}\cos 4n\alpha + \&c)}{\sinh n\beta\,(1-2e^{-2\beta^2}\cosh 2n\beta + e^{-4\beta^2})(1-2e^{-4\beta^2}\cosh 4n\beta + \&c)}$$

$$= e^{\frac{\alpha^2-\beta^2}{6}}.$$

12. If $\alpha\beta = \pi$

$$\left\{\frac{\alpha^2}{12} + \frac{\cos 2n\alpha}{1(e^{2\alpha^2}-1)} + \frac{\cos 4n\alpha}{2(e^{4\alpha^2}-1)} + \frac{\cos 6n\alpha}{3(e^{6\alpha^2}-1)} + \&c\right\}$$

$$-\left\{\frac{\beta^2}{12} + \frac{\cosh 2n\beta}{1(e^{2\beta^2}-1)} + \frac{\cosh 4n\beta}{2(e^{4\beta^2}-1)} + \frac{\cosh 6n\beta}{3(e^{6\beta^2}-1)} + \&c\right\}$$

$$= \frac{n^2}{2} + \frac{1}{2}\log_e\frac{\sin n\alpha}{\sinh n\beta}.$$

13. If $\alpha\beta = \pi$

$$\frac{n}{2} + \frac{\alpha\sinh 2n\alpha}{e^{2\alpha^2}-1} + \frac{\alpha\sinh 4n\alpha}{e^{4\alpha^2}-1}$$

$$\frac{1^2 x}{(1-x)^2} + \frac{2^2 x^2}{(1-x^2)^2} + \frac{3^2 x^3}{(1-x^3)^2} + \frac{4^2 x^4}{(1-x^4)^2} + \&c$$

$$= \frac{1}{288}\left\{1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} + \&c\right)\right\}$$

$$- \frac{1}{288}\left\{1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right)\right\}^2$$

$$\frac{1^4 x}{(1-x)^2} + \frac{2^4 x^2}{(1-x^2)^2} + \frac{3^4 x^3}{(1-x^3)^2} + \frac{4^4 x^4}{(1-x^4)^2} + \&c$$

$$= \frac{1}{720}\left\{1 + 240\left(\frac{x}{1-x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} + \&c\right)\right\}$$

$$\times \left\{1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right)\right\}$$

$$- \frac{1}{720}\left\{1 - 504\left(\frac{x}{1-x} + \frac{2^5 x^2}{1-x^2} + \frac{3^5 x^3}{1-x^3} + \&c\right)\right\}.$$

$$\frac{1^6 x}{(1-x)^2} + \frac{2^6 x^2}{(1-x^2)^2} + \frac{3^6 x^3}{(1-x^3)^2} + \frac{4^6 x^4}{(1-x^4)^2} +$$

$$= \frac{1}{1008}\left\{1 + 480\left(\frac{x}{1-x} + \frac{2^7 x^2}{1-x^2} + \frac{3^7 x^3}{1-x^3} + \&c\right)\right\}$$

$$- \frac{1}{1008}\left\{1 - 504\left(\frac{x}{1-x} + \frac{2^5 x^2}{1-x^2} + \frac{3^5 x^3}{1-x^3} + \&c\right)\right\}$$

$$\times \left\{1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right)\right\}$$

$$\frac{1^8 x}{(1-x)^2} + \frac{2^8 x^2}{(1-x^2)^2} + \&c$$

$$= \frac{1}{720}\left\{1 + 480\left(\frac{x}{1-x} + \frac{2^7 x^2}{1-x^2} + \frac{3^7 x^3}{1-x^3} + \&c\right)\right\}$$

$$\times \left\{1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right)\right\}$$

$$- \frac{1}{720}\left\{1 - 264\left(\frac{x}{1-x} + \frac{2^9 x^2}{1-x^2} + \frac{3^9 x^3}{1-x^3} + \&c\right)\right\},$$

$$+\frac{\beta\sin 2n\beta}{e^{2\beta^2}-1}+\frac{\beta\sin 4n\beta}{e^{4\beta^2}-1}+\frac{\beta\sin 6n\beta}{e^{6\beta^2}-1}+\&c.$$

$$=\frac{\alpha\coth n\alpha-\beta\cot n\beta}{4}.$$

14. If $\alpha\beta=\pi^2$ and n is a positive integer >1

$$\alpha^n\left\{\frac{B_{2n}}{4n}\cos\pi n+\frac{1^{2n-1}}{e^{2\alpha}-1}+\frac{2^{2n-1}}{e^{4\alpha}-1}+\frac{3^{2n-1}}{e^{6\alpha}-1}+\&c\right\}$$

$$=(-\beta)^n\left\{\frac{B_{2n}}{4n}\cos\pi n+\frac{1^{2n-1}}{e^{2\beta}-1}+\frac{2^{2n-1}}{e^{4\beta}-1}+\frac{3^{2n-1}}{e^{6\beta}-1}+\&c\right\}$$

Cor. 1. $\frac{1^5}{e^{2\pi}-1}+\frac{2^5}{e^{4\pi}-1}+\frac{3^5}{e^{6\pi}-1}+\frac{4^5}{e^{8\pi}-1}+\&c=\frac{1}{504}$

2. $\frac{1^9}{e^{2\pi}-1}+\frac{2^9}{e^{4\pi}-1}+\frac{3^9}{e^{6\pi}-1}+\frac{4^9}{e^{8\pi}-1}+\&c=\frac{1}{264}$

3. $\frac{1^{13}}{e^{2\pi}-1}+\frac{2^{13}}{e^{4\pi}-1}+\frac{3^{13}}{e^{6\pi}-1}+\frac{4^{13}}{e^{8\pi}-1}+\&c=\frac{1}{24}$

15. If $\alpha\beta=\pi^2$ and n any integer

$$(4\alpha)^{1-n}\left\{\frac{1}{2}S_{2n-1}+\frac{1}{1^{2n-1}(e^{2\alpha}-1)}+\frac{1}{2^{2n-1}(e^{4\alpha}-1)}+\&c\right\}$$

$$-(-4\beta)^{1-n}\left\{\frac{1}{2}S_{2n-1}+\frac{1}{1^{2n-1}(e^{2\beta}-1)}+\frac{1}{2^{2n-1}(e^{4\beta}-1)}+\&c\right\}$$

$$=\frac{B_{2n}}{\lfloor\underline{2n}}\left\{(-\alpha)^n+\beta^n\right\}+\pi^2\frac{B_2}{\lfloor\underline{2}}\cdot\frac{B_{2n-2}}{\lfloor\underline{2n-2}}\left\{(-\alpha)^{n-2}+\beta^{n-2}\right\}$$

$$-\pi^4\frac{B_4}{\lfloor\underline{4}}\cdot\frac{B_{2n-4}}{\lfloor\underline{2n-4}}\left\{(-\alpha)^{n-4}+\beta^{n-4}\right\}+\&c$$ the last term

being $-\pi^n\frac{B_n}{\lfloor\underline{n}}\cdot\frac{B_n}{\lfloor\underline{n}}(-1)^{\frac{n}{2}}$ or $\pi^{n-1}\frac{B_{n-1}}{\lfloor\underline{n-1}}\cdot\frac{B_{n+1}}{\lfloor\underline{n+1}}(-1)^{\frac{n+1}{2}}\{(-\alpha)+$ [illegible]

according as n is even or odd.

If $P_n = \frac{B_n}{2n}(2^n-1)\cos\frac{\pi n}{2} + \frac{1^{n-1}x}{1+x} - \frac{2^{n-1}x^2}{1+x^2} + \frac{3^{n-1}x^3}{1+x^3} - \&c$

& $Q_n = \dfrac{\frac{1}{2}E_{n+1}\cos\frac{\pi n}{2} + \frac{1^n x}{1-x} - \frac{3^n x^3}{1-x^3} + \frac{5^n x^5}{1-x^5} - \&c}{\frac{1}{2}E_1 + \frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \&c}$

then $\frac{1}{2}Q_n = 2^n P_n - \frac{(n-1)(n-2)}{\lfloor\underline{2}} 2^{n-2} P_{n-2} Q_2$

$+ \frac{(n-1)(n-2)(n-3)(n-4)}{\lfloor\underline{4}} 2^{n-4} P_{n-4} Q_4 - \&c$

If $P_n = \frac{B_n}{2n}\cos\frac{\pi n}{2} + \frac{1^{n-1}x}{1-x} + \frac{2^{n-1}x^2}{1-x^2} + \frac{3^{n-1}x^3}{1-x^3} + \&c$

& $Q_n = \frac{1}{n+1}\cdot\dfrac{1^{n+1} - 3^{n+1}x + 5^{n+1}x^3 - 7^{n+1}x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 +}$

then $\frac{1}{2}Q_n = -2^n P_n - \frac{(n-1)(n-2)}{\lfloor\underline{2}} 2^{n-2} P_{n-2} Q_2 -$

$\frac{(n-1)(n-2)(n-3)(n-4)}{\lfloor\underline{4}} 2^{n-4} P_{n-4} Q_4 - \&c$

$\log_e \dfrac{(1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12})\&c}{1 - 2x\cos\theta + 2x^4\cos 2\theta - 2x^9\cos 3\theta + \&c}$

$= 2\left\{\frac{x}{1-x^2}\cos\theta + \frac{x^2}{2(1-x^4)}\cos 2\theta + \frac{x^3}{3(1-x^6)}\cos 3\theta + \&c\right\}$

Cor. 1. $\frac{1}{1^3(e^{2\pi}-1)} + \frac{1}{2^3(e^{4\pi}-1)} + \frac{1}{3^3(e^{6\pi}-1)} + \&c = \frac{7\pi^3}{360} - \frac{S_3}{2}$.

Cor. 2 $\frac{1}{1^7(e^{2\pi}-1)} + \frac{1}{2^7(e^{4\pi}-1)} + \frac{1}{3^7(e^{6\pi}-1)} + \&c = \frac{19\pi^7}{113400} - \frac{S_7}{2}$.

16. If $\alpha\beta = \pi$, $f(n) = \int_0^\infty e^{-x^2}\phi(nx)\,dx$ and $f(n)f(ni) = f(0)f(0)$, then

$f(0)\sqrt{\alpha}\left\{\frac{1}{2}\phi(0) + e^{-\alpha^2}\phi(n\alpha) + e^{-4\alpha^2}\phi(2n\alpha) + e^{-9\alpha^2}\phi(3n\alpha) + \&c\right\}$

$= f(n)\sqrt{\beta}\left\{\frac{1}{2}\phi(0) + e^{-\beta^2}\phi(n\beta i) + e^{-4\beta^2}\phi(2n\beta i) + e^{-9\beta^2}\phi(3n\beta i) + \&c\right\}$

17. $1 + \frac{2\cos nx}{1+x^2} + \frac{2\cos 2nx}{1+4x^2} + \frac{2\cos 3nx}{1+9x^2} + \&c$

$= \frac{\pi}{x}\coth\frac{\pi}{x}\cosh n - \frac{\pi}{x}\sinh n$

if n lies between 0 and $\frac{2\pi}{x}$.

18. $\phi(0) + \frac{\phi(xi) + \phi(-xi)}{1+x^2} + \frac{\phi(2xi) + \phi(-2xi)}{1+4x^2} + \&c$

$= \frac{\pi}{2x}\coth\frac{\pi}{x}\left\{\phi(1) + \phi(-1)\right\} + \frac{\pi}{2x}\left\{\phi(1) - \phi(-1)\right\}$.

N.B. Within certain limits of x only this theorem is true; so we must be very careful in applying this theorem. e.g.

$1 + \frac{2\cos mx\cos nx}{1+x^2} + \frac{2\cos 2mx\cos 2nx}{1+4x^2} + \&c$

$= \frac{\pi}{2x}\coth\frac{\pi}{x}\cosh m\cosh n - \sinh m\cosh n$

is true only when $m+n$ & $m-n$ lies between 0 and $\frac{2\pi}{x}$.

We may write $\phi\{f(x) + \psi(x)\}$ for $\phi(x)$.

This theorem is always true when $x = $ ...

$$\log_e \frac{1+4\left(\frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \frac{x^7}{1-x^7} + \&c\right)}{1+4\cos n\left(\frac{x\cos n}{1-x} - \frac{x^3\cos 3n}{1-x^3} + \frac{x^5\cos 5n}{1-x^5} - \&c\right)}$$

$$= 4\left\{\frac{x\sin^2 n}{1(1+x)} - \frac{x^2\sin^2 2n}{2(1+x^2)} + \frac{x^3\sin^2 3n}{3(1+x^3)} - \&c\right\}$$

$$\frac{1}{4}\log_e \frac{\sin n - x\sin 3n + x^3\sin 5n - x^6\sin 7n + \&c}{\sin n(1 - 3x + 5x^3 - 7x^6 + 9x^{10} - \&c)}$$

$$= 4\left\{\frac{x\sin^2 n}{1(1-x)} + \frac{x^2\sin^2 2n}{2(1-x^2)} + \frac{x^3\sin^2 3n}{3(1-x^3)} + \&c\right\}.$$

$$\frac{1}{4}\log_e \frac{\sin 2n - x\sin 4n + x^5\sin 8n - x^8\sin 10n + \&c}{\sin 2n(1 - 2x + 4x^5 - 5x^8 + 7x^{16} - \&c)}$$

$$= \frac{x}{1+x}\sin^2 n + \frac{x^2\sin^2 2n}{2(1+x^2)} + \frac{x^3\sin^2 3n}{3(1+x^3)} + \&c$$

$$+ \frac{x^4}{1-x^4}\sin^2 2n + \frac{x^8}{2(1-x^8)}\sin^2 4n + \&c$$

$$\frac{1}{4}\log_e \frac{\sin n - x\sin 5n + x^2\sin 7n - x^5\sin 11n + \&c}{\sin n(1 - 5x + 7x^2 - 11x^5 + 13x^7 - \&c)}$$

$$= \frac{x\sin^2 n}{1-x} + \frac{x^2\sin^2 2n}{2(1-x^2)} + \frac{x^3\sin^2 3n}{3(1-x^3)} + \&c$$

$$+ \frac{x\sin^2 2n}{1+x} + \frac{x^2\sin^2 4n}{2(1+x^2)} + \frac{x^3\sin^2 6n}{3(1+x^3)} + \&c.$$

19 If $\alpha\beta = \pi$, then

$$\left\{\frac{\alpha^2}{6}\phi(0) + \frac{\alpha x}{2}\phi'(0) + \frac{x^2}{4}\phi''(0)\right\}$$

$$+\left\{\frac{\phi(x\alpha)+\phi(-x\alpha)}{1(e^{2\alpha^2}-1)} + \frac{\phi(2x\alpha)+\phi(-2x\alpha)}{2(e^{4\alpha^2}-1)} + \frac{\phi(3x\alpha)+\phi(-3x\alpha)}{3(e^{6\alpha^2}-1)} + \&c\right\}$$

$$+\left\{\phi(x\alpha) + \tfrac{1}{2}\phi(2x\alpha) + \tfrac{1}{3}\phi(3x\alpha) + \&c\right\}$$

$$= \left\{\frac{\beta^2}{6}\phi(0) + \frac{\beta x i}{2}\phi'(0) + \frac{\alpha\beta i}{2}\phi(0)\right\}$$

$$+\left\{\frac{\phi(x\beta i)+\phi(-x\beta i)}{1(e^{2\beta^2}-1)} + \frac{\phi(2x\beta i)+\phi(-2x\beta i)}{2(e^{4\beta^2}-1)} + \frac{\phi(3x\beta i)+\phi(-3x\beta i)}{3(e^{6\beta^2}-1)} + \&c\right\}$$

$$+\left\{\phi(x\beta i) + \tfrac{1}{2}\phi(2x\beta i) + \tfrac{1}{3}\phi(3x\beta i) + \&c\right\}$$

20. If $\alpha\beta = \pi$, m any even integer and $\psi(m) = \frac{B_m}{|\underline{m}}\left\{(\alpha i)^m + \beta^m\right\}$

$+ \frac{B_2}{|\underline{2}}\cdot\frac{B_{m-2}}{|\underline{m-2}}(\alpha\beta)^2\left\{(\alpha i)^{m-4} + \beta^{m-4}\right\} - \frac{B_4}{|\underline{4}}\cdot\frac{B_{m-4}}{|\underline{m-4}}(\alpha\beta)^4\left\{(\alpha i)^{m-8} + \beta^{m-8}\right\}$

$+ \&c$ the last term being $-(\alpha\beta i)^{\frac{m}{2}-1}\frac{B_{\frac{m}{2}-1}}{|\underline{\frac{m}{2}-1}}\cdot\frac{B_{\frac{m}{2}+1}}{|\underline{\frac{m}{2}+1}}\left\{(\alpha i)^2 + \beta^2\right\}$

or $-(\alpha\beta i)^{\frac{m}{2}}\frac{B_{\frac{m}{2}}}{|\underline{\frac{m}{2}}}\cdot\frac{B_{\frac{m}{2}}}{|\underline{\frac{m}{2}}}$ according as $\frac{m}{2}$ is odd or even, then

$$\alpha\left\{\frac{\phi(2x\alpha)+\phi(-2x\alpha)}{(2\alpha)^{m-1}(e^{2\alpha^2}-1)} + \frac{\phi(4x\alpha)+\phi(-4x\alpha)}{(4\alpha)^{m-1}(e^{4\alpha^2}-1)} + \&c\right\} + \alpha\left\{\frac{\phi(2x\alpha)}{(2\alpha)^{m-1}} + \frac{\phi(4x\alpha)}{(4\alpha)^{m-1}} \ldots\right.$$

$$\mp\beta i\left\{\frac{\phi(2x\beta i)+\phi(-2x\beta i)}{(2\beta i)^{m-1}(e^{2\beta^2}-1)} + \frac{\phi(4x\beta i)+\phi(-4x\beta i)}{(4\beta i)^{m-1}(e^{4\beta^2}-1)} + \&c\right\}$$

$$+\frac{\beta i}{2}\left\{\frac{\phi(2x\beta i)+\phi(-2x\beta i)}{(2\beta i)^{m-1}} + \frac{\phi(4x\beta i)+\phi(-4x\beta i)}{\ldots}\right\}$$

$$+\frac{\alpha}{2}\left\{\frac{B_{m-2}}{|\underline{m-2}}\beta^{m-2}\frac{x}{|\underline{1}}\phi'(0) + \frac{B_{m-4}}{|\underline{m-4}}\beta^{m-4}\frac{x^3}{|\underline{3}}\phi'''(0) + \ldots - \frac{x^{m-1}}{|\underline{m-1}}\frac{\phi^{m-1}(0)}{2}\right.$$

$$+\phi(0)\psi(m) + \frac{x^2}{|\underline{2}}\phi''(0)\psi(m-2) + \ldots + \frac{x^m}{|\underline{m}}\phi^m(0)\psi(0)$$

This theorem is always true whatever ...

$$\frac{\phi'(x)}{\phi(x)} - \frac{\psi'(x)}{\psi(x)} = \frac{1-\phi^4(-x)}{8x}$$

Show that

$$\frac{\psi'(x)}{\psi(x)} - 2x\frac{\psi'(x^2)}{\psi(x^2)} = \frac{1-\phi^4(-x)}{8x} \qquad \therefore \frac{\phi'(x)}{\phi(x)} - \frac{\phi'(-x)}{\phi(-x)} = \frac{\phi^4(x)-\phi^4(-x)}{4x}.$$

$$\frac{1^3 - 3^3x + 5^3x^3 - 7^3x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c}$$
$$= 1 - 24\left\{\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right\}$$

$$4\,\frac{1^3 - 3^3x^4 + 5^3x^{12} - 7^3x^{24} + \&c}{1 - 3x^4 + 5x^{12} - 7x^{24} + \&c} - \frac{1^3 - 3^3x + 5^3x^3 - 7^3x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c} = 3\phi^4(x)$$

$$\frac{1}{240}\cdot\frac{1^5 - 3^5x + 5^5x^3 - 7^5x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c}$$
$$= \frac{1}{240} + \frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \&c$$
$$- 2\left\{\frac{x}{(1-x)^2} + \frac{4x^2}{(1-x^2)^2} + \frac{9x^3}{(1-x^3)^2} + \&c\right\}$$

1. $\left(\frac{1}{x-n+1} - \frac{1}{x-n+3} + \frac{1}{x-n+5} - \&c\right)$

$-\left(\frac{1}{x+n+1} - \frac{1}{x+n+3} + \frac{1}{x+n+5} - \&c\right)$

$= \frac{n}{x^2-1+}\,\frac{2^2-n^2}{1+}\,\frac{2^2}{x^2-1+}\,\frac{4^2-n^2}{1+}\,\frac{4^2}{x^2-1+\&c}$

2. $\phi(0) - \frac{1}{2}\phi(1) + \frac{1.3}{2.4}\phi(2) - \&c$

$= \phi(-\frac{1}{2}) - \frac{1}{2}\phi(1\frac{1}{2}) + \frac{1.3}{2.4}\phi(-2\frac{1}{2}) - \&c$

3. $\frac{1}{4}\phi^2(x) = \frac{1}{4} + \frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \frac{x^7}{1-x^7} + \&c$

4. $\frac{1}{8}\phi^4(x) = \frac{1}{8} + \frac{x}{1-x} + \frac{2x^2}{1+x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1+x^4} + \&c$

5. If $\alpha\beta = \pi^2$, then

$\sqrt{\alpha}\left\{\frac{1}{4} + \frac{1}{e^{\alpha}-1} - \frac{1}{e^{3\alpha}-1} + \frac{1}{e^{5\alpha}-1} - \&c\right\}$

$= \sqrt{\beta}\left\{\frac{1}{4} + \frac{1}{e^{\beta}-1} - \frac{1}{e^{3\beta}-1} + \frac{1}{e^{5\beta}-1} - \&c\right\}$

6. If $\int_0^\infty \phi(x)\cos nx\,dx = \psi(n)$, then

$\int_0^\infty \psi(x)\cos nx\,dx = \frac{\pi}{2}\phi(n)$.

7. If $\int_0^\infty \phi(x)\cos nx\,dx = \psi(n)$, then

(i) $\int_0^\infty \psi(x)\,dx = \frac{\pi}{2}\phi(0)$, (ii) $\int_0^\infty \psi^2(x)\,dx = \frac{\pi}{2}\int_0^\infty \phi^2(x)\,dx$

8. If $\int_0^\infty \phi(x)\cos nx\,dx = \psi(n)$, then

(i) $\int_0^\infty x^r\phi(x)\cos nx\cos\frac{\pi r}{2}\,dx = \psi^r(n)$

(ii) $\int_0^\infty x^r\psi(x)\cos nx\cos\frac{\pi r}{2}\,dx = \frac{\pi}{2}\phi^r(n)$.

$$1 - 5x + 7x^2 - 11x^5 + 13x^7 - \&c = \phi^2(-x) f(-x, -x^2) = B_x$$

$$f(-x, -x^2) f(-x^2, -x^4) = \phi(-x) \psi(x)$$

$$\frac{f(-x, -x^4)}{f(-x^4, -x^8)} = \frac{\phi(-x^2)}{\psi(x)}$$

$$1 - 2x + 4x^5 - 5x^8 + 7x^{16} - \&c = \psi(x^2) f^2(-x, -x^2) = A_x$$

Show that $\frac{1}{4}\phi^2(x)\phi^2(x^2) =$

$$\frac{1}{4} + \frac{x}{1-x} + \frac{x^2}{1+x^2} + \frac{3x^3}{1-x^3} + \frac{6x^4}{1+x^4} + \frac{5x^5}{1-x^5} + \frac{3x^6}{1+x^6} \ldots$$

$$\frac{1}{2}\phi(0) + \phi(1) + \phi(2) + \phi(3) + \&c$$

$$= \int_0^\infty \phi(x) - \frac{\phi(xi) - \phi(-xi)}{i(e^{2\pi x} - 1)}\, dx.$$

$$A_x + 2x A_{x^4} = B_{x^8}$$

$$1 - 5(x^3) - 7(x^3)^2 + 11(x^3)^5 + 13(x^3)^7 - \&c$$

$$= \phi^2(x)\psi(-x) + 3x\phi^2(x^9).\psi(-x^9).$$

$$\frac{(1-x)^5(1-x^2)^5(1-x^3)^5(1-x^4)^5 \&c}{(1-x^5)(1-x^{10})(1-x^{15})(1-x^{20}) \&c}$$

$$= 1 - 5\left(\frac{x}{1+x} - \frac{3x^3}{1+x^3} + \frac{4x^4}{1+x^4} - \frac{7x^7}{1+x^7} + \frac{9x^9}{1+x^9} \ldots + \frac{11x^{11}}{1+x^{11}} - \frac{12x^{12}}{1+x^{12}}\right)$$

9. $\phi(x)\,\phi(x^2) = 1 + \frac{2x}{1-x} + \frac{2x^3}{1-x^3} - \frac{2x^5}{1-x^5} - \frac{2x^7}{1-x^7} + \&c$

10. $\phi(x)\,\phi(x^3) = 1 + \frac{2x}{1-x} - \frac{2x^2}{1+x^2} + \frac{2x^4}{1+x^4} - \frac{2x^5}{1-x^5} + \frac{2x^7}{1-x^7} - \&c$

show that

11. $\phi(x)\,\phi(x^6) = 1 + \frac{2x}{1-x} - \frac{2x^2}{1+x^2} - \frac{2x^3}{1-x^3} + \frac{2x^5}{1-x^5} + \frac{2x^6}{1+x^6} - \frac{2x^7}{1-x^7} + \frac{2x^9}{1-x^9} - \frac{2x^{10}}{1+x^{10}}\ \&c$

12. $1 - 3x + 5x^3 - 7x^6 + 9x^{10} - \&c$
$= \{(1-x)(1-x^2)(1-x^3)(1-x^4)\ \&c\}^3 = \phi^2(-x)\,\psi(x).$

13. If $\alpha\beta = \frac{\pi}{4}$, then
$\sqrt{\alpha}\,\{\alpha e^{-\alpha^2} - 3\alpha e^{-9\alpha^2} + 5\alpha e^{-25\alpha^2} - \&c\}$
$= \sqrt{\beta}\,\{\beta e^{-\beta^2} - 3\beta e^{-9\beta^2} + 5\beta e^{-25\beta^2} - \&c\}$

14. If $\int_0^\infty \phi(x)\cos 2nx\,dx = \psi(n)$, then
$\alpha\{\tfrac{1}{2}\phi(0) + \phi(\alpha)\cos 2n\alpha + \phi(2\alpha)\cos 4n\alpha + \phi(3\alpha)\cos 6n\alpha + \&c\}$
$= \psi(n) + \psi(\beta - n) + \psi(\beta + n) + \psi(2\beta - n) + \psi(2\beta + n) + \&c$
with $\alpha\beta = \pi$ & n lying between 0 & β.

N. B. $\int_0^\infty \frac{e^{ax} - e^{-ax}}{e^{\pi x} - e^{-\pi x}} \cos mx\,dx = \frac{\sin a}{e^m + 2\cos a + e^{-m}}$

$\int_0^\infty \frac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}} \sin mx\,dx = \frac{1}{2}\,\frac{e^m - e^{-m}}{e^m + 2\cos a + e^{-m}}$

$\int_0^\infty \frac{\sin mx}{e^{2\pi x} - 1}\,dx = \frac{1}{2}\left(\frac{1}{e^m - 1} + \frac{1}{2} - \frac{1}{m}\right)$. $\int_0^\infty \frac{\phi(ax) - \phi(bx)}{x}\,dx$

~~$= \log \frac{b}{a}$~~. $= \log\frac{a}{b}\,\{\phi(\infty) - \phi(0)\}$

If $y = \pi \dfrac{1+\left(\frac{1}{2}\right)^2(1-x)+\left(\frac{1\cdot3}{2\cdot4}\right)^2(1-x)^2+\cdots}{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1\cdot3}{2\cdot4}\right)^2x^2+\cdots}$

then $\dfrac{dy}{dx} = -\dfrac{1}{x(1-x)\left\{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1\cdot3}{2\cdot4}\right)^2x^2+\cdots\right\}^2}$

$$\left(\tfrac{1}{2}\right)^2 + 2\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x + 3\left(\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2x^2+\cdots$$

$$= \left\{\frac{1}{3(1-x)} - \frac{1}{12x}\right\}\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\cdots\right\}$$

$$+ \frac{1-24\left(\frac{1}{e^y-1}+\frac{2}{e^{2y}-1}+\frac{3}{e^{3y}-1}+\cdots\right)}{12x(1-x)\left\{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1\cdot3}{2\cdot4}\right)^2x^2+\cdots\right\}}$$

$$1+240\left(\frac{1^3}{e^y-1}+\frac{2^3}{e^{2y}-1}+\frac{3^3}{e^{3y}-1}+\cdots\right)$$

$$= \left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\cdots\right\}^4(1+14x+x^2)$$

$$1-504\left(\frac{1^5}{e^y-1}+\frac{2^5}{e^{2y}-1}+\frac{3^5}{e^{3y}-1}+\cdots\right)$$

$$= \left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\cdots\right\}^6(1+x)(1-34x+x^2)$$

If x is changed to $\left(\dfrac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2$ then

y is changed to $2y$

$$1 + 240\left(\frac{1^3}{e^{y}-1} + \frac{2^3}{e^{2y}-1} + \&c\right)$$

$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c\right\}^4 (1 - x + x^2)$$

$$1 - 504\left(\frac{1^5}{e^{y}-1} + \frac{2^5}{e^{2y}-1} + \&c\right)$$

$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c\right\}^6 (1+x)\left(1-\tfrac{x}{2}\right)(1-2x).$$

$$1 - 8\left(\frac{1}{e^{y}+1} - \frac{2}{e^{2y}+1} + \frac{3}{e^{3y}+1} - \&c\right)$$

$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c\right\}^2 (1-x)$$

$$1 + 16\left(\frac{1^3}{e^{y}+1} - \frac{2^3}{e^{2y}+1} + \frac{3^3}{e^{3y}+1} - \&c\right)$$

$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c\right\}^4 (1-x^2)$$

$$1 - 8\left(\frac{1^5}{e^{y}+1} - \frac{2^5}{e^{2y}+1} + \frac{3^5}{e^{3y}+1} - \&c\right)$$

$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c\right\}^6 (1-x)(1-x+x^2).$$

$$17 + 32\left(\frac{1^7}{e^{y}+1} - \frac{2^7}{e^{2y}+1} + \frac{3^7}{e^{3y}+1} - \&c\right)$$

$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c\right\}^8 (1-x)(17-32x+x^2)$$

$$1 + 24\left(\frac{1}{e^{y}+1} + \frac{2}{e^{2y}+1} + \frac{3}{e^{3y}+1} + \&c\right)$$

$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c\right\}^2 (1+x)$$

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$$1 + 4\left(\frac{1}{e^{y}+e^{-y}} + \frac{1}{e^{2y}+e^{-2y}} + \&c\right)$$

$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c\right\}.$$

$$4\left(\frac{1^2}{e^{y}+e^{-y}} + \frac{2^2}{e^{2y}+e^{-2y}} + \frac{3^2}{e^{3y}+e^{-3y}} + \&c\right)$$

$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c\right\}^3 \left(\frac{x}{4}\right).$$

$$4\left(\frac{1^4}{e^{y}+e^{-y}} + \frac{2^4}{e^{2y}+e^{-2y}} + \frac{3^4}{e^{3y}+e^{-3y}} + \&c\right)$$

$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c\right\}^5 \left\{\left(\frac{x}{4}\right) + \left(\frac{x}{4}\right)^2\right\}$$

$$4\left(\frac{1^6}{e^{y}+e^{-y}} + \frac{2^6}{e^{2y}+e^{-2y}} + \frac{3^6}{e^{3y}+e^{-3y}} + \&c\right)$$

$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c\right\}^7 \left\{\left(\frac{x}{4}\right) + 11\left(\frac{x}{4}\right)^2 + \left(\frac{x}{4}\right)^3\right\}.$$

$$4\left(\frac{1^8}{e^{y}+e^{-y}} + \frac{2^8}{e^{2y}+e^{-2y}} + \frac{3^8}{e^{3y}+e^{-3y}} + \&c\right)$$

$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot3}{2\cdot4}\right)^2 x^2 + \right\}^9 \left\{\left(\frac{x}{4}\right) + 57\left(\frac{x}{4}\right)^2 + 102\left(\frac{x}{4}\right)^3 + \left(\frac{x}{4}\right)^4\right\}$$

If $\alpha\beta = \pi^2$, then

$$\alpha\left\{\frac{\sec n\alpha}{4} + \frac{\cos n\alpha}{e^{\alpha^2}-1} - \frac{\cos 3n\alpha}{e^{3\alpha^2}-1} + \frac{\cos 5n\alpha}{e^{5\alpha^2}-1} - \&c\right\}$$

$$= \beta\left\{\frac{1}{4} + \frac{\cosh 2n\beta}{e^{\beta^2}+e^{-\beta^2}} + \frac{\cosh 4n\beta}{e^{2\beta^2}+e^{-2\beta^2}} + \&c\right\}$$

$$1+4\left(\frac{1}{e^{y}-1}-\frac{1}{e^{3y}-1}+\frac{1}{e^{5y}-1}-\&c\right)$$

$$=\left\{1+\left(\frac{1}{2}\right)^2 x+\left(\frac{1.3}{2.4}\right)^2 x^2+\&c\right\}.$$

$$1-4\left(\frac{1^2}{e^{y}-1}-\frac{3^2}{e^{3y}-1}+\frac{5^2}{e^{5y}-1}-\&c\right)$$

$$=\left\{1+\left(\frac{1}{2}\right)^2 x+\left(\frac{1.3}{2.4}\right)^2 x^2+\&c\right\}^3(1-x).$$

$$5+4\left(\frac{1^4}{e^{y}-1}-\frac{3^4}{e^{3y}-1}+\frac{5^4}{e^{5y}-1}-\&c\right)$$

$$=\left\{1+\left(\frac{1}{2}\right)^2 x+\left(\frac{1.3}{2.4}\right)^2 x^2+\&c\right\}^5(5-x)(1-x).$$

$$61-4\left(\frac{1^6}{e^{y}-1}-\frac{3^6}{e^{3y}-1}+\frac{5^6}{e^{5y}-1}-\&c\right)$$

$$=\left\{1+\left(\frac{1}{2}\right)^2 x+\left(\frac{1.3}{2.4}\right)^2 x^2+\&c\right\}^7(1-x)(61-46x+x^2)$$

$$1+240\left(\frac{1^3x}{1-x}+\frac{2^3x^2}{1-x^2}+\frac{3^3x^3}{1-x^3}+\&c\right)$$

$$=\phi^8(-x)+256x\,\psi^8(x).$$

Similarly it is possible to express any identity in terms of $\phi(x)$ & $\psi(x)$.

$$\int\frac{\phi^{2n+2}(u)}{u}\,du=\int\left\{1+\left(\frac{1}{2}\right)^2 x+\left(\frac{1.3}{2.4}\right)^2 x^2+\&c\right\}^n\frac{dx}{x(1-x)}$$

when $x=1-\dfrac{\phi^4(-u)}{\phi^4(u)}$.

or $u=F(x)$ or e^{-y}.

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$$1-16\left(\frac{1^3}{e^y-1}-\frac{2^3}{e^{2y}-1}+\frac{3^3}{e^{3y}-1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1.3}{2.4}\right)^2x^2+\&c\right\}^4(1-x)^2.$$

$$1+8\left(\frac{1^5}{e^y-1}-\frac{2^5}{e^{2y}-1}+\frac{3^5}{e^{3y}-1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1.3}{2.4}\right)^2x^2+\&c\right\}^6(1-x)(1-x^2).$$

$$17-32\left(\frac{1^7}{e^y-1}-\frac{2^7}{e^{2y}-1}+\frac{3^7}{e^{3y}-1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1.3}{2.4}\right)^2x^2+\&c\right\}^8(1-x)^2(17-2x+17x^2)$$

$$31+8\left(\frac{1^9}{e^y-1}-\frac{2^9}{e^{2y}-1}+\frac{3^9}{e^{3y}-1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1.3}{2.4}\right)^2x^2+\&c\right\}^{10}(1-x)(1-x^2)(31-46x+31x^2).$$

$$\frac{1^3}{e^y-e^{-y}}+\frac{2^3}{e^{2y}-e^{-2y}}+\frac{3^3}{e^{3y}-e^{-3y}}+\&c$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1.3}{2.4}\right)^2x^2+\right\}^4\frac{x}{16}$$

$$\frac{1^5}{e^y-e^{-y}}+\frac{2^5}{e^{2y}-e^{-2y}}+\frac{3^5}{e^{3y}-e^{-3y}}+\&c$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1.3}{2.4}\right)^2x^2+\&c\right\}^6\frac{x(1+x)}{16}$$

$$\frac{1^7}{e^y-e^{-y}}+\frac{2^7}{e^{2y}-e^{-2y}}+\frac{3^7}{e^{3y}-e^{-3y}}+\&c$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1.3}{2.4}\right)^2x^2+\&c\right\}^8\frac{x(1+6\tfrac{1}{2}x+x^2)}{16}.$$

$$\frac{1^9}{e^y-e^{-y}}+\frac{2^9}{e^{2y}-e^{-2y}}+\frac{3^9}{e^{3y}-e^{-3y}}+\&c$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1.3}{2.4}\right)^2x^2+\&c\right\}^{10}\frac{x(1+x)(1+29x+x^2)}{16}.$$

$$1 + 24\left(\frac{1}{e^{y}+1} + \frac{2}{e^{2y}+1} + \frac{3}{e^{3y}+1} + \&c\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}^2 (1+x).$$

$$1 - 240\left(\frac{1^3}{e^{y}+1} + \frac{2^3}{e^{2y}+1} + \frac{3^3}{e^{3y}+1} + \&c\right)$$

$$= \underbrace{\left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}}_{z}^4 (1 - 16x + x^2).$$

$$1 + 504\left(\frac{1^5}{e^{y}+1} + \frac{2^5}{e^{2y}+1} + \frac{3^5}{e^{3y}+1} + \&c\right)$$

$$= z^6 (1+x)(1+29x+x^2).$$

$$1 + 24\left(\frac{1}{e^{2y}+1} + \frac{2}{e^{4y}+1} + \frac{3}{e^{6y}+1} + \&c\right)$$

$$= z^2\left(1 - \frac{x}{2}\right).$$

$$1 - 16\left(\frac{1^3}{e^{2y}-1} - \frac{2^3}{e^{4y}-1} + \frac{3^3}{e^{6y}-1} - \&c\right)$$

$$= z^4 (1-x).$$

$$1 + 8\left(\frac{1^5}{e^{2y}-1} - \frac{2^5}{e^{4y}-1} + \frac{3^5}{e^{6y}-1} - \&c\right)$$

$$= z^6 (1-x)\left(1 - \frac{x}{2}\right).$$

$$17 - 32\left(\frac{1^7}{e^{2y}-1} - \frac{2^7}{e^{4y}-1} + \frac{3^7}{e^{6y}-1} - \&c\right)$$

$$= z^8 (1-x)(17 - 17x + 2x^2)$$

$$1 + 240\left(\frac{1^3}{e^{4y}-1} + \frac{2^3}{e^{8y}-1} + \frac{3^3}{e^{12y}-1} + \&c\right) = z^4\left(1 - x + \frac{x}{16}\right.$$

$$1 - 504\left(\frac{1^5}{e^{4y}-1} + \frac{2^5}{e^{8y}-1} + \frac{3^5}{e^{12y}-1} + \&c\right) = z^6\left(1 - \frac{x}{2}\right)\left(1 - x - \frac{x}{32}\right.$$

If $\alpha\beta = \pi^2$, then

$$\frac{\sqrt{\alpha}}{\alpha^n}\left\{\frac{1}{2\alpha}\left(\frac{1}{1^{2n}}-\frac{1}{3^{2n}}+\frac{1}{5^{2n}}-\&c\right)+\frac{1}{1^{2n}(e^{\alpha}-1)}-\frac{1}{3^{2n}(e^{3\alpha}-1)}+\frac{1}{5^{2n}(e^{5\alpha}-1)}-\frac{1}{7^{2n}(e^{7\alpha}-1)}+\&c\right\}$$

$$=\frac{\sqrt{\beta}}{\beta^n}\left\{(-1)^n\left[\frac{1}{2^{2n}(e^{\beta}+e^{-\beta})}+\frac{1}{4^{2n}(e^{2\beta}+e^{-2\beta})}+\&c\right]\right.$$

$$+\frac{1}{4}\left[\frac{\left(\frac{\beta}{2}\right)^{2n}}{\lfloor 2n}E_{2n+1}+\frac{B_2}{\lfloor 2}\cdot\frac{E_{2n-1}}{\lfloor 2n-2}\cdot\left(\frac{\beta}{2}\right)^{2n-1}(2\alpha)\right.$$

$$\left.\left.-\frac{B_4}{\lfloor 4}\cdot\frac{E_{2n-3}}{\lfloor 2n-4}\left(\frac{\beta}{2}\right)^{2n-2}(2\alpha)^2+\&c\ \&c-\frac{(-\alpha\beta)^n}{\lfloor 2n}B_{2n}E_1\right]\right\}$$

$$\log_e\frac{1+\frac{4x}{1+x^2}+\frac{4x^2}{1+x^4}+\frac{4x^3}{1+x^6}+\&c}{1+\frac{4x\cos 2x}{1+x^2}+\frac{4x^2\cos 4x}{1+x^4}+\&c}$$

$$=8\left\{\frac{x\sin^2 x}{1(1-x^2)}+\frac{x^3\sin^2 3x}{3(1-x^6)}+\&c\right\}.$$

$$1+\frac{4x\cos x}{1+x^2}+\frac{4x^2\cos 2x}{1+x^4}+\frac{4x^3\cos 3x}{1+x^6}+\&c$$

$$=\frac{(-x^4)^3\{1-2x^2\cos 2x+2x^8\cos 4x-2x^{18}\cos 6x+\&c\}}{(1-2x\cos x+2x^4\cos 2x-2x^9\cos 3x+\&c)^2}$$

$$1 + \frac{4x\cos n}{1+x^2} + \frac{4x^4\cos 2n}{1+x^4} + \frac{4x^9 \cos 3n}{1+x^6} + \&c$$

$$\phi^2(-x^2) \cdot \frac{1+2x\cos n + 2x^4\cos 2n + 2x^9\cos 3n + \&c}{1-2x\cos n + 2x^4\cos 2n - 2x^9\cos 3n + \&c}$$

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$$\frac{1}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}} - \frac{3}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}} + \frac{5}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}} - \&c$$

$$= \frac{z^2}{4}\sqrt{x(1-x)}.$$

$$\frac{1^3}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}} - \frac{3^3}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}} + \frac{5^3}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}} - \&c$$

$$= \frac{z^4}{4}\sqrt{x(1-x)}\,(1-2x).$$

$$\frac{1^5}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}} - \frac{3^5}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}} + \frac{5^5}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}} - \&c$$

$$= \frac{z^6}{4}\sqrt{x(1-x)}\,(1-16x+16x^2).$$

$$\frac{1^7}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}} - \frac{3^7}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}} + \frac{5^7}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}} - \&c$$

$$= \frac{z^8}{4}\sqrt{x(1-x)}\,(1-2x)(1-136x+136x^2).$$

$$\frac{1^9}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}} - \frac{3^9}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}} + \frac{5^9}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}} - \&c$$

$$= \frac{z^{10}}{4}\sqrt{x(1-x)}\left\{1-1232x(1-x)+7936x^2(1-x)^2\right\}$$

the last term $= 2^{2n}(2^n-1)\dfrac{B_n}{n}$.

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If $\alpha\beta = \pi^2$ and n a positive integer then

$$\alpha^{2n+1}\left\{\frac{1^{2n+1}}{e^{\frac{\alpha}{2}}+e^{-\frac{\alpha}{2}}} - \frac{3^{2n+1}}{e^{\frac{3\alpha}{2}}+e^{-\frac{3\alpha}{2}}} + \frac{5^{2n+1}}{e^{\frac{5\alpha}{2}}+e^{-\frac{5\alpha}{2}}} - \&c\right\}$$

$$\mp(-\beta)^{2n+1}\left\{\frac{1^{2n+1}}{e^{\frac{\beta}{2}}+e^{-\frac{\beta}{2}}} - \frac{3^{2n+1}}{e^{\frac{3\beta}{2}}+e^{-\frac{3\beta}{2}}} + \frac{5^{2n+1}}{e^{\frac{5\beta}{2}}+e^{-\frac{5\beta}{2}}} - \&c\right\}$$

$$= 0$$

If $\alpha\beta = \pi$, then

$$\alpha\left\{\frac{\phi(\alpha)-\phi(-\alpha)}{e^{\frac{\alpha^2}{2}}+e^{-\frac{\alpha^2}{2}}} - \frac{\phi(3\alpha)-\phi(-3\alpha)}{e^{\frac{3\alpha^2}{2}}+e^{-\frac{3\alpha^2}{2}}} + \&c\right\}$$

$$\mp\beta i\left\{\frac{\phi(\beta i)-\phi(-\beta i)}{e^{\frac{\beta^2}{2}}+e^{-\frac{\beta^2}{2}}} - \frac{\phi(3\beta i)-\phi(-3\beta i)}{e^{\frac{3\beta^2}{2}}+e^{-\frac{3\beta^2}{2}}} + \&c\right\}$$

$$= 0$$

If $\alpha\beta = \pi$, then

$$\alpha\left\{\frac{\sin n\alpha}{e^{\frac{\alpha^2}{2}}+e^{-\frac{\alpha^2}{2}}} - \frac{\sin 3n\alpha}{e^{\frac{3\alpha^2}{2}}+e^{-\frac{3\alpha^2}{2}}} + \&c\right\}$$

$$= \beta\left\{\frac{\sinh n\beta}{e^{\frac{\beta^2}{2}}+e^{-\frac{\beta^2}{2}}} - \frac{\sinh 3n\beta}{e^{\frac{3\beta^2}{2}}+e^{-\frac{3\beta^2}{2}}} + \&c\right\}$$

If $\alpha\beta = \pi^2$, then

$$\frac{1}{1(e^{\frac{\alpha}{2}}+e^{-\frac{\alpha}{2}})} - \frac{1}{3(e^{\frac{3\alpha}{2}}+e^{-\frac{3\alpha}{2}})} + \frac{1}{5(e^{\frac{5\alpha}{2}}+e^{-\frac{5\alpha}{2}})} - \&c$$

$$+ \frac{1}{1(e^{\frac{\beta}{2}}+e^{-\frac{\beta}{2}})} - \frac{1}{3(e^{\frac{3\beta}{2}}+e^{-\frac{3\beta}{2}})} + \frac{1}{5(e^{\frac{5\beta}{2}}+e^{-\frac{5\beta}{2}})} - \&c$$

$$= \frac{\pi}{8}.$$

If $\alpha\beta = \frac{\pi^2}{4}$, then

$$(\tan^{-1} e^{-\alpha} - \tan^{-1} e^{-3\alpha} + \tan^{-1} e^{-5\alpha} - \&c)$$
$$+(\tan^{-1} e^{-\beta} - \tan^{-1} e^{-3\beta} + \tan^{-1} e^{-5\beta} - \&c) = \frac{\pi}{8}$$

$$\tan^{-1} e^{-\frac{y}{2}} - \tan^{-1} e^{-\frac{3y}{2}} + \tan^{-1} e^{-\frac{5y}{2}} - \&c$$

$$= \frac{1}{4} \sin^{-1} \sqrt{x}$$

$$\log_e(1-e^{-2y}) + \log_e(1-e^{-4y}) + \log_e(1-e^{-6y}) +$$

$$= \frac{1}{12}\left\{y + \log_e \frac{x(1-x)}{16}\right\} + \frac{1}{2}\log_e\left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \cdots\right\}$$

$$1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)$$

$$= (1-2x)z^2 + 6x(1-x)z\,\frac{dz}{dx}$$

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$$\frac{1}{e^y-e^{-y}}+\frac{3}{e^{3y}-e^{-3y}}+\frac{5}{e^{5y}-e^{-5y}}+\&c=\frac{x}{16}z^2.$$

$$\frac{1^3}{e^y-e^{-y}}+\frac{3^3}{e^{3y}-e^{-3y}}+\frac{5^3}{e^{5y}-e^{-5y}}+\&c$$

$$=\frac{x}{16}z^4\left(1-\frac{x}{2}\right)$$

$$\frac{1^5}{e^y-e^{-y}}+\frac{3^5}{e^{3y}-e^{-3y}}+\frac{5^5}{e^{5y}-e^{-5y}}+\&c$$

$$=\frac{x}{16}z^6\left(1-x+x^2\right)$$

$$\frac{1^7}{e^y-e^{-y}}+\frac{3^7}{e^{3y}-e^{-3y}}+\frac{5^7}{e^{5y}-e^{-5y}}+\&c$$

$$=\frac{x}{16}z^8\left(1-\frac{x}{2}\right)\left(1-x+\frac{17}{2}x^2\right).$$

$$\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\&c$$

$$=x\frac{1+x}{(1-x)^2}-x^3\frac{1+x^2}{(1-x^2)^2}+x^6\frac{1+x^3}{(1-x^3)^2}-\&c$$

$$\frac{1+x}{1-x}-x\cdot\frac{1+x^3}{1-x^3}+x^3\cdot\frac{1+x^5}{1-x^5}-x^6\cdot\frac{1+x^7}{1-x^7}+\&c$$

$$=\psi(x)\,\phi(x^2)$$

$$\frac{1+x}{1-x}-x^2\cdot\frac{1+x^3}{1-x^3}+x^6\cdot\frac{1+x^5}{1-x^5}-x^{12}\cdot\frac{1+x^7}{1-x^7}+\&c$$

$$=\psi^2(x).$$

If $\alpha\beta = \pi^2$, then

$$\frac{\pi^2}{12} + \frac{1}{1^2(e^\alpha - e^{-\alpha})^2} + \frac{1}{2^2(e^{2\alpha} - e^{-2\alpha})^2} + \frac{1}{3^2(e^{3\alpha} - e^{-3\alpha})^2} + \&c$$

$$+ \frac{1}{1^2(e^\beta - e^{-\beta})^2} + \frac{1}{2^2(e^{2\beta} - e^{-2\beta})^2} + \frac{1}{3^2(e^{3\beta} - e^{-3\beta})^2} + \&c$$

$$- 2\alpha\left\{1^2\log_e(1-e^{-2\alpha}) + 2^2\log_e(1-e^{-4\alpha}) + 3^2\log_e(1-e^{-6\alpha}) + \&c\right\}$$

$$- 2\beta\left\{1^2\log_e(1-e^{-2\beta}) + 2^2\log_e(1-e^{-4\beta}) + 3^2\log_e(1-e^{-6\beta}) + \&c\right\}$$

$$= \frac{\alpha^2+\beta^2}{120}.$$

If $\alpha\beta = \pi^2$ and n any integer, then

$$\alpha^{1-n}\left\{\frac{1}{1^{2n-1}(e^{\frac{\alpha}{2}} + e^{-\frac{\alpha}{2}})} - \frac{1}{3^{2n-1}(e^{\frac{3\alpha}{2}} + e^{-\frac{3\alpha}{2}})} + \&c\right\}$$

$$+(-\beta)^{1-n}\left\{\frac{1}{1^{2n-1}(e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}})} - \frac{1}{3^{2n-1}(e^{\frac{3\beta}{2}} + e^{-\frac{3\beta}{2}})} + \&c\right\} =$$

$$= \frac{\pi}{2^{2n+1}}\left[\frac{E_1 E_{2n-1}}{\lfloor 2n-2}\left\{(-\alpha)^{n-1} + \beta^{n-1}\right\} - \frac{E_3 E_{2n-3}}{\lfloor 2 \lfloor 2n-4} \times \left\{(-\alpha)^{n-3} + \beta^{n-3}\right\} + \frac{E_5 E_{2n-5}}{\lfloor 4 \lfloor 2n-6}\left\{(-\alpha)^{n-5} + \beta^{n-5}\right\} - \&c\right]$$

the last term being $(-1)^{\frac{n-1}{2}}\left(\frac{E_n}{\lfloor n-1}\right)^2$ or $(-1)^{\frac{n}{2}}\frac{E_{n-1}E_{n+1}}{\lfloor n-2 \lfloor n}(\alpha-\beta)$ according as n is odd or ev

$$\frac{1}{1^2(e^{\frac{y}{2}}+e^{-\frac{y}{2}})}+\frac{1}{3^2(e^{\frac{3y}{2}}+e^{-\frac{3y}{2}})}+\frac{1}{5^2(e^{\frac{5y}{2}}+e^{-\frac{5y}{2}})}+\&c$$

$$=\frac{\sqrt{x}}{4}\cdot\frac{1+\left(\frac{2}{3}\right)^2x+\left(\frac{2.4}{3.5}\right)^2x^2+\left(\frac{2.4.6}{3.5.7}\right)^2x^3+\&c}{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\left(\frac{1.3.5}{2.4.6}\right)^2x^3+\&c}$$

$$\frac{1}{1^2(e^{\frac{y}{2}}-e^{-\frac{y}{2}})}-\frac{1}{3^2(e^{\frac{3y}{2}}-e^{-\frac{3y}{2}})}+\frac{1}{5^2(e^{\frac{5y}{2}}-e^{-\frac{5y}{2}})}-\&c$$

$$=\frac{\sqrt{x}}{4}\cdot\frac{1+\left(\frac{2}{3}\right)^2\left\{1+\left(\frac{1}{2}\right)^2\right\}x+\left(\frac{2.4}{3.5}\right)^2\left\{1+\left(\frac{1}{2}\right)^2+\left(\frac{1.3}{2.4}\right)^2\right\}x^2+\&c}{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\left(\frac{1.3.5}{2.4.6}\right)^2x^3+\&c}$$

$$\frac{1}{1^4(e^{\frac{y}{2}}+e^{-\frac{y}{2}})}+\frac{1}{3^4(e^{\frac{3y}{2}}+e^{-\frac{3y}{2}})}+\frac{1}{5^4(e^{\frac{5y}{2}}+e^{-\frac{5y}{2}})}+\&c$$

$$=\frac{\sqrt{x}}{4}\cdot\frac{1+\left(\frac{2}{3}\right)^2\left[1+\left(\frac{2}{3}\right)^2\left\{1+\left(\frac{1}{2}\right)^2\right\}\right]x+\&c}{\left\{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\&c\right\}^3}$$

$$\sqrt{1-x}+\left(\frac{2}{3}\right)^2\sqrt{1-x}^{\,3}+\left(\frac{2.4}{3.5}\right)^2\sqrt{1-x}^{\,5}+$$

$$=\frac{\pi^2}{4}\left\{1+\left(\frac{1}{2}\right)^2(1-x)+\left(\frac{1.3}{2.4}\right)^2(1-x)^2+\right\}$$

$$-3\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\&c\right)\left\{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\right\}$$

$$+4\left\{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\right\}\left\{\frac{1}{1^2(e^y+1)}-\frac{1}{3^2(e^{3y}+1)}\ldots\right.$$

$$\frac{1}{1(e^{\frac{y}{2}}-e^{-\frac{y}{2}})}+\frac{1}{3(e^{\frac{3y}{2}}-e^{-\frac{3y}{2}})}+\frac{1}{5(e^{\frac{5y}{2}}-e^{-\frac{5y}{2}})}+\&c$$

$$=\frac{\sqrt{x}}{4}\left(1+\frac{x}{3}+\frac{x^2}{5}+\frac{x^3}{7}+\&c\right)$$

$$\frac{1}{1(e^{\frac{y}{2}}+e^{-\frac{y}{2}})}-\frac{1}{3(e^{\frac{3y}{2}}+e^{-\frac{3y}{2}})}+\frac{1}{5(e^{\frac{5y}{2}}+e^{-\frac{5y}{2}})}-\&c$$

$$=\frac{\sqrt{x}}{4}\left\{1+\frac{1}{2}\cdot\frac{x}{3}+\frac{1.3}{2.4}\cdot\frac{x^2}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^3}{7}+\&c\right\}.$$

If $P_1e^{-y}+P_2e^{-2y}+P_3e^{-3y}+\&c$

$$=Q_1x+Q_2x^2+Q_3x^3+\&c=f(x).$$

then $P_1e^{-y}-P_2e^{-2y}+P_3e^{-3y}-\&c$

$$=Q_1\left(\frac{x}{1-x}\right)-Q_2\left(\frac{x}{1-x}\right)^2+Q_3\left(\frac{x}{1-x}\right)^3-\&c=-f\left(-\frac{x}{1-x}\right).$$

If $F(e^{-y})=f(x)$ then $F(-e^{-y})=f\left(-\frac{x}{1-x}\right)$.

$$1-\frac{4x}{1+x}+\frac{4x^3}{1+x^2}-\frac{4x^6}{1+x^3}+\frac{4x^{10}}{1+x^4}-\&c$$

$$=(1-2x+2x^4-2x^9+2x^{16}-\&c)^2$$

$$\frac{\psi^3(x)}{\psi(x^3)}=1+3\left(\frac{x}{1-x}-\frac{x^5}{1-x^5}+\frac{x^7}{1-x^7}-\frac{x^{11}}{1-x^{11}}+\&c\right)$$

$$\frac{\phi^3(x)}{\phi(x^3)}=1+6\left(\frac{x}{1-x}+\frac{x^2}{1+x^2}-\frac{x^4}{1+x^4}-\frac{x^5}{1-x^5}+\&c\right)$$

$$\left.\begin{aligned}\frac{\phi^3(x)}{\phi(x^3)}+2\,\frac{\phi^3(-x^2)}{\phi(-x^6)}&=3\,\phi(x)\,\phi(x^3)\\ \frac{\phi^3(x)}{\phi(x^3)}+\frac{\phi^3(-x^2)}{\phi(-x^6)}&=2\,\frac{\psi^3(x)}{\psi(x^3)}\end{aligned}\right\}$$

$$F\left(\frac{2-\sqrt{3}}{4}\right)=e^{-\pi\sqrt{3}}$$

If $$3.\ \frac{1+\left(\frac{1}{2}\right)^2(1-\alpha)+\left(\frac{1.3}{2.4}\right)^2(1-\alpha)^2+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\left(\frac{1.3}{2.4}\right)^2\alpha^2+\&c}=\frac{1+\left(\frac{1}{2}\right)^2(1-\beta)+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\&c}$$

then $$\sqrt[8]{\frac{\alpha^3}{\beta}}-\sqrt[8]{\frac{(1-\alpha)^3}{1-\beta}}=1.$$

and $$\sqrt{\sqrt[8]{\frac{\alpha^3}{\beta}}-\alpha}\left\{1+\left(\frac{1}{2}\right)^2\alpha+\left(\frac{1.3}{2.4}\right)^2\alpha^2+\&c\right\}$$

$$=\sqrt{\sqrt[8]{\frac{\alpha^3}{\beta}}-\beta}\left\{1+\left(\frac{1}{2}\right)^2\beta+\left(\frac{1.3}{2.4}\right)^2\beta^2+\&c\right\}$$

and $$\sqrt[8]{\frac{\alpha^3}{\beta}}+\sqrt[8]{\frac{(1-\alpha)^3}{1-\beta}}=3\sqrt{\frac{\sqrt[8]{\frac{\alpha^3}{\beta}}-\alpha}{\sqrt[8]{\frac{\alpha^3}{\beta}}-\beta}}$$

If $\sin^3\theta = m^4 \sin\phi$ & $\cos^3\theta = n^4\cos\phi$
and $m - n = 1$

then $\sqrt{\dfrac{m-\sin^2\theta}{m-\sin^2\phi}} = \dfrac{m+n}{3}$.

$$= \frac{1+\left(\frac{1}{2}\right)^2\sin^2\phi+\left(\frac{1.3}{2.4}\right)^2\sin^4\phi+}{1+\left(\frac{1}{2}\right)^2\sin^2\theta+\left(\frac{1.3}{2.4}\right)^2\sin^4\theta+}$$

If $\dfrac{\alpha^3}{\beta} = (1+p)^8$ & $\dfrac{(1-\alpha)^3}{1-\beta} = p^8$

then $\alpha = \dfrac{(1+p)^2(1-p^2)}{1+2p}$

& $\beta = \dfrac{(1-p)^2(1-p^2)}{(1+2p)^3}$.

$$1-\alpha = p^3\cdot\frac{p+2}{1+2p}.$$

$$1-\beta = p\cdot\left(\frac{p+2}{1+2p}\right)^3.$$

$$\left\{1+\left(\tfrac{1}{2}\right)^2 p\left(\frac{2+p}{1+2p}\right)^3+\left(\tfrac{1.3}{2.4}\right)^2p^2\left(\frac{2+p}{1+2p}\right)^6+\&c\right\}$$

$$=(1+2p)\left\{1+\left(\tfrac{1}{2}\right)^2p^3\cdot\frac{2+p}{1+2p}+\left(\tfrac{1.3}{2.4}\right)^2p^6\left(\frac{2+p}{1+2p}\right)\ldots\right\}$$

If $p^3\cdot\dfrac{2+p}{1+2p} = \dfrac{1-\sqrt{1-x^3}}{2}$

then $p = \dfrac{-(1+\sqrt{1-x})+\sqrt{2+x+2\sqrt{1+\ldots}}}{2}$

If $\sqrt[8]{\alpha} = m$ & $\sqrt[8]{\beta} = n$

$$m^4+2m^3n^3-\ldots-n^4=0$$

$$F\left\{\frac{1}{2}-(2-\sqrt{3})\sqrt[4]{12}\right\} = e^{-3\pi}$$

$$\phi(e^{-3\pi}) = \frac{\phi(e^{-\pi})}{\sqrt[4]{6\sqrt{3}-9}}$$

$$1+\frac{n}{1}\cdot\frac{m}{2m}\cdot x\cdot\left(\frac{2+x}{1+2x}\right)^3+\&c$$

$$=(1+2x)^{2n}\left\{1+\frac{1}{2}\cdot\frac{n}{1}\cdot x^3\cdot\frac{2+x}{1+2x}+\&c\right\} \text{ if } m=\frac{2n+1}{6}$$

$$x\frac{\psi^3(x^3)}{\psi(x)} = \frac{x}{1-x^2}-\frac{x^2}{1-x^4}+\frac{x^4}{1-x^8}-\frac{x^5}{1-x^{10}}+\frac{x^7}{1-x^{14}}$$
$$-\frac{x^8}{1-x^{16}}+\frac{x^{10}}{1-x^{20}}-\&c$$

$$\frac{\phi^3(x^3)}{\phi(x)} = 1-2\left(\frac{x}{1+x}-\frac{x^2}{1-x^2}+\frac{x^4}{1-x^4}-\frac{x^5}{1+x^5}+\frac{x^7}{1+x^7}\right.$$
$$\left.-\frac{x^8}{1-x^8}+\frac{x^{10}}{1-x^{10}}-\frac{x^{11}}{1+x^{11}}+\&c\right)$$

$$\frac{\psi^3(-x^3)}{\psi(-x)}-\frac{\psi^3(x^3)}{\psi(x)} = 2x\frac{\psi^3(x^6)}{\psi(x^2)}$$

$$\frac{\psi^5(-x^5)}{\psi(-x)}-\frac{\psi^5(x^5)}{\psi(x)} = 4x^3\frac{\psi^5(x^{10})}{\psi(x^4)}$$
$$+2x\frac{f^5(-x^{20},-x^{40})}{f(-x^4,-x^8)}$$

$$\sqrt[8]{\frac{(1-\beta)^3}{1-\alpha}}-\sqrt[8]{\frac{\beta^3}{\alpha}}=1$$

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$$\frac{\frac{1}{2}+e^{-\pi x}\cos(\pi\sqrt{1-x^2})+e^{-4\pi x}\cos(4\pi\sqrt{1-x^2})+\&c}{e^{-\pi x}\sin(\pi\sqrt{1-x^2})+e^{-4\pi x}\sin(4\pi\sqrt{1-x^2})+\&c}$$

$$=\frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{1-x}}.$$

$$(\sqrt{5}+\sqrt{3})\left\{1+2e^{-\frac{\pi\sqrt{5}}{3}}+2e^{-\frac{4\pi\sqrt{5}}{3}}+2e^{-\frac{9\pi\sqrt{5}}{3}}+\&c\right\}$$

$$=(3+\sqrt{3})\left\{1+2e^{-3\pi\sqrt{5}}+2e^{-12\pi\sqrt{5}}+2e^{-27\pi\sqrt{5}}+\&c\right\}$$

$$\frac{1}{\phi^4(e^{-\pi})}=.71777$$

$$\phi(e^{-5\pi})=\frac{\phi(e^{-\pi})}{\sqrt{5\sqrt{5}-10}}$$

$$\frac{\psi^3(x)}{\psi(x^3)}+\frac{\psi^3(-x)}{\psi(-x^3)}=2.\frac{\psi^3(x^2)}{\psi(x^6)}.$$

$$\frac{\psi^5(x)}{\psi(x^5)}+\frac{\psi^5(-x)}{\psi(-x^5)}+12\frac{f^5(-x^4,-x^8)}{f(-x^{20},-x^{40})}$$

$$=4\frac{\psi^5(x^2)}{\psi(x^{10})}.$$

$$\sqrt[8]{\frac{(1-\beta)^5}{1-\alpha}}-\sqrt[8]{\frac{\beta^5}{\alpha}}=1+\sqrt[3]{2}\cdot\sqrt[24]{\frac{\beta^2(1-\beta)^2}{\alpha(1-\alpha)}}$$

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If 5. $$\frac{1+\left(\frac{1}{2}\right)^2(1-\alpha)+\left(\frac{1.3}{2.4}\right)^2(1-\alpha)^2+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\left(\frac{1.3}{2.4}\right)^2\alpha^2+\&c}$$

$$=\frac{1+\left(\frac{1}{2}\right)^2(1-\beta)+\left(\frac{1.3}{2.4}\right)^2(1-\beta)^2+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\left(\frac{1.3}{2.4}\right)^2\beta^2+\&c}$$

then $$\sqrt[8]{\frac{\alpha^5}{\beta}}-\sqrt[8]{\frac{(1-\alpha)^5}{1-\beta}}=1+\sqrt[3]{2}\cdot\sqrt[24]{\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}}$$

If 9. $$\left.\begin{array}{l}\dfrac{1+\left(\frac{1}{2}\right)^2(1-\alpha)+\left(\frac{1.3}{2.4}\right)^2(1-\alpha)^2+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\left(\frac{1.3}{2.4}\right)^2\alpha^2+\&c}\\ =\dfrac{1+\left(\frac{1}{2}\right)^2(1-\beta)+\left(\frac{1.3}{2.4}\right)^2(1-\beta)^2+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\left(\frac{1.3}{2.4}\right)^2\beta^2+\&c}\end{array}\right\}=3\cdot\frac{1+\left(\frac{1}{2}\right)^2(1-\gamma)+}{1+\left(\frac{1}{2}\right)^2\gamma+}$$

then $$\sqrt[8]{\frac{\alpha}{\beta}}\sqrt[4]{\frac{1-\alpha}{1-\beta}}\left\{1+\left(\tfrac{1}{2}\right)^2\alpha+\left(\tfrac{1.3}{2.4}\right)^2\alpha^2+\&c\right\}$$
$$+3\left\{1+\left(\tfrac{1}{2}\right)^2\beta+\left(\tfrac{1.3}{2.4}\right)^2\beta^2+\&c\right\}$$
$$=\left(\sqrt[8]{\frac{\alpha}{\beta}}+\sqrt[4]{\frac{1-\alpha}{1-\beta}}\right)\sqrt{\left\{1+\left(\tfrac{1}{2}\right)^2\alpha+\&c\right\}\left\{1+\left(\tfrac{1}{2}\right)^2\beta+\&c\right\}}$$

$$\frac{\phi^5(-x)}{\phi(-x^5)}+4\cdot\frac{(1-x)^5(1-x^2)^5(1-x^3)^5\cdots}{(1-x^5)(1-x^{10})(1-x^{15})\cdots}=5\,\phi^3(-x)\,\phi(-x^5)$$

$$1 + 2\sqrt[8]{\frac{\beta^3}{\alpha}} \quad \& \quad 1 + 2\sqrt[3]{2}\sqrt[24]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}}$$

$$\times\, 1 + 2\sqrt[8]{\frac{(1-\alpha)^3}{1-\beta}} \qquad \times\, 1 + 2\sqrt[3]{2}\sqrt[24]{\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}}$$

$$= 3 \qquad\qquad = 5$$

$$F\left\{\frac{1-\sqrt{1-(2-\sqrt{3})^4}}{2}\right\} = e^{-3\pi}$$

$$F\left\{\frac{1-\sqrt{1-(\sqrt{5}-2)^8}}{2}\right\} = e^{-5\pi}$$

$$F\left(\frac{1}{2}-\sqrt{\sqrt{5}-2}\right) = e^{-\pi\sqrt{5}}$$

$$F\left(\frac{8-3\sqrt{7}}{16}\right) = e^{-\pi\sqrt{7}}$$

$$1 + \left(\frac{1}{2}\right)^2 p\left(\frac{2-p}{1+2p}\right)^5 + \left(\frac{1.5}{2.8}\right)^2 p^2\left(\frac{2-p}{1+2p}\right)^{10} + \&c$$

$$= (1+2p)\left\{1 + \left(\frac{1}{2}\right)^2 p^5\,\frac{2-p}{1+2p} + \left(\frac{1.5}{2.8}\right)^2 p^{10}\left(\frac{2-p}{1+2p}\right)^2 + \right\}$$

$$\phi(e^{-9\pi}) = \frac{1+\sqrt[3]{2(\sqrt{3}+1)}}{3}\,\phi(e^{-\pi})$$

$$1 + \left(\frac{1}{2}\right)^2 \frac{1 - \frac{1-11p-p^2}{(1+2p)^2}\sqrt{\frac{1+p^2}{1+2p}}}{2} + \&c$$

$$= (1+2p)\left\{1 + \left(\frac{1}{2}\right)^2 \frac{1-(1+p-p^2)\sqrt{\frac{1+p^2}{1+2p}}}{2} + \&c\right\}$$

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$$\frac{\psi^7(x)}{\psi(x^7)} + \frac{\psi^7(-x)}{\psi(-x^7)} + 14 \cdot \frac{f^7(-x^8,-x^{16})}{f(-x^{56},-x^{112})}$$

$$= 8\frac{\psi^7(x^2)}{\psi(x^{14})} + 2\frac{f^7(-x^2,-x^4)}{f(-x^{14},-x^{28})} + 6\frac{f^7(-x^4,-x^8)}{f(-x^{28},-x^{56})}$$

$$8\frac{\psi^7(x^2)}{\psi(x^{14})} + 2\frac{f^7(-x^2,-x^6)}{f(-x^{14},-x^{48})} - \frac{\psi^7(x)}{\psi(x^7)} - \frac{\psi^7(-x)}{\psi(-x^7)}$$

is a function of x^4

$$\frac{\left\{\frac{\phi^2(x)}{\phi(x^7)} - \phi^2(x^7)\right\}^2 \frac{\phi^3(x^7)}{\phi(x)} - \left\{\phi^2(-x) - \phi^2(-x^7)\right\}^2 \frac{\phi^3(-x^7)}{\phi(-x)}}{x^3}$$

is function of x^4.

$$\phi(x^{14}) = \phi(x^2)\, e^{i\left\{\frac{\pi}{4} - \tan^{-1}\frac{\phi(-x)}{\phi(x)}\right\}}$$

$$\phi^3(x^2)\,\phi^3(x^{14}) \sin\left\{3\tan^{-1}\frac{\phi(-x^7)}{\phi(x^7)} - 3\tan^{-1}\frac{\phi(-x)}{\phi(x)}\right\}$$

$$+ 2\phi(x^2)\,\phi^5(x^{14}) \sin\left\{5\tan^{-1}\frac{\phi(-x^7)}{\phi(x^7)} - \tan^{-1}\frac{\phi(-x)}{\phi(x)}\right\} + 1 =$$

$$+ \frac{\phi^7(x^{14})}{\phi(x^2)} \sin\left\{7\tan^{-1}\frac{\phi(-x^7)}{\phi(x^7)} + \tan^{-1}\frac{\phi(-x)}{\phi(x)}\right\}$$

$$F\left(\frac{\sqrt{6}-\sqrt{2}-1}{\sqrt{6}+\sqrt{2}+1}\right) = e^{-\pi\sqrt{6}}$$

$$\frac{1-2\beta}{2\alpha-1} = \frac{f(x)}{f\left(\frac{n}{x}\right)}$$

$$F\frac{(7-4\sqrt{3})(4-\sqrt{15})}{(\sqrt{5}+1)^4} = e^{-\pi\sqrt{15}}.$$

$$F\left(\frac{16-7\sqrt{3}\pm\sqrt{15}}{32}\right) = e^{-\pi\sqrt{15}}$$

$$F\left\{(2-\sqrt{3})^4(\sqrt{2}-1)^6\right\} = e^{-3\pi\sqrt{2}}$$

$$F\left(\frac{\sqrt{6}-\sqrt{2}-1}{\sqrt{2}-1}\right)^2 = e^{-\pi\sqrt{6}}$$

7. $$\frac{1+\left(\frac{1}{2}\right)^2(1-\alpha)+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\&c} = \frac{1+\left(\frac{1}{2}\right)^2(1-\beta)+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\&c}$$

~~VII~~ $\sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)} = 1$

III $\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)} = 1$

I. $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} = 1$

$$\sqrt{7.\ \frac{\sqrt[8]{\alpha} - \beta}{\alpha - \sqrt[8]{\alpha\beta}}}$$

V $\sqrt[8]{\alpha\beta^3} + \sqrt[8]{(1-\alpha)(1-\beta)^3}$

$= \sqrt[8]{\alpha^3\beta} + \sqrt[8]{(1-\alpha)^3(1-\beta)}$

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III. $\sqrt[8]{\alpha\beta^5} + \sqrt[8]{(1-\alpha)(1-\beta)^5}$

$= \sqrt[8]{\alpha^5\beta} + \sqrt[8]{(1-\alpha)^5(1-\beta)}$

III ✓ $\dfrac{1-2\sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}}}{1-2\sqrt[4]{\alpha\beta}} = \sqrt{1+4\sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}}} = \dfrac{1+(\frac{1}{2})^2\alpha+}{1+(\frac{1}{2})^2\beta}$

VII + $\dfrac{1-4\sqrt[24]{\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}}}{1-2\sqrt[8]{\alpha\beta}} = \dfrac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}$

V $\dfrac{1+\sqrt[8]{\frac{(1-\beta)^5}{1-\alpha}}}{1+\sqrt[8]{(1-\alpha)^3(1-\beta)}} = \dfrac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+} = \dfrac{1-\sqrt[8]{\frac{\beta^5}{\alpha}}}{1-\sqrt[8]{\alpha^3\beta}}$

III $\dfrac{-1+\sqrt[4]{\frac{(1-\beta)^3}{1-\alpha}}}{1-\sqrt[4]{(1-\alpha)(1-\beta)}} = \dfrac{1-\sqrt[4]{\frac{\beta^3}{\alpha}}}{1-\sqrt[4]{\alpha\beta}} = \dfrac{1+(\frac{1}{2})^2\alpha+}{1+(\frac{1}{2})^2\beta+}$

VII $\sqrt{7\,\dfrac{\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}}-\sqrt[8]{\frac{\beta^7}{\alpha}}}{\sqrt[8]{\frac{\alpha^7}{\beta}}-\sqrt[8]{\frac{(1-\alpha)^7}{1-\beta}}}} = \dfrac{1+(\frac{1}{2})^2\alpha+}{1+(\frac{1}{2})^2\beta+}$

III $\sqrt{3\cdot\dfrac{1+\sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}}}{1+\sqrt[8]{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}}}} = \dfrac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}$

VII $\sqrt{7\cdot\dfrac{1-4\sqrt[24]{[illegible]}}{-1}}$ [illegible]

III $\sqrt{3}\cdot\dfrac{1-2\sqrt[8]{\dfrac{\beta^2(1-\beta)^2}{\alpha(1-\alpha)}}}{2\sqrt[8]{\dfrac{\alpha^2(1-\alpha)^2}{\beta(1-\beta)}}-1} = \dfrac{1+\left(\frac{k}{l}\right)^2\alpha + [illegible]}{1+\left(\frac{k}{l}\right)^2\beta + [illegible]}$

Solvable from above

XI $\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}+2\sqrt[3]{2}\,\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}=1$

XXIII $\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}+\sqrt[3]{4}\,\sqrt[24]{\alpha\beta(1-\alpha)(1-\beta)}=1$

IV $\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}+2\sqrt[3]{4}\,\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}=1$

VII. $\sqrt{\dfrac{\beta}{\alpha}}+\sqrt{\dfrac{1-\beta}{1-\alpha}}-\sqrt{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}=\sqrt{\left(\dfrac{1+\left(\frac{k}{l}\right)^2\alpha+[illegible]}{1+\left(\frac{k}{l}\right)^2\beta+[illegible]}\right)^2}$

V $\sqrt[4]{\dfrac{\beta}{\alpha}}+\sqrt[4]{\dfrac{1-\beta}{1-\alpha}}-\sqrt[4]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}=\dfrac{1+\left(\frac{k}{l}\right)^2\alpha+[illegible]}{1+\left(\frac{k}{l}\right)^2\beta+[illegible]}$

IX $\sqrt[8]{\dfrac{\beta}{\alpha}}+\sqrt[8]{\dfrac{1-\beta}{1-\alpha}}-\sqrt[8]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}=\sqrt{\dfrac{1+\left(\frac{k}{l}\right)^2\alpha+[illegible]}{1+\left(\frac{k}{l}\right)^2\beta+[illegible]}}$

XXI $\sqrt[8]{\dfrac{\beta}{\alpha}}+\sqrt[8]{\dfrac{1-\beta}{1-\alpha}}-\sqrt[8]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}-2\sqrt[12]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}=\sqrt{[illegible]}$

XIII $\sqrt[4]{\dfrac{\beta}{\alpha}}+\sqrt[4]{\dfrac{1-\beta}{1-\alpha}}-\sqrt[4]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}-4\sqrt[6]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}=\dfrac{1+\left(\frac{k}{l}\right)^2\alpha+[illegible]}{1+\left(\frac{k}{l}\right)^2\beta+[illegible]}$

VII $\sqrt{\dfrac{\beta}{\alpha}}+\sqrt{\dfrac{1-\beta}{1-\alpha}}-\sqrt{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}-8\sqrt[3]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}=\left\{1+\left(\frac{k}{l}\right)^2\alpha\ [illegible]\right.$

XV $\sqrt[16]{\alpha\beta}\left(\sqrt[4]{1+\sqrt{\alpha}}\,\sqrt[4]{1+\sqrt{\beta}}+\sqrt[4]{1-\sqrt{\alpha}}\,\sqrt[4]{1-\sqrt{\beta}}\right)$

$+\sqrt[16]{(1-\alpha)(1-\beta)}\left(\sqrt[4]{1+\sqrt{1-\alpha}}\,\sqrt[4]{1+\sqrt{1-\beta}}+\sqrt[4]{1-\sqrt{1-\alpha}}\,\sqrt[4]{1-\sqrt{1-\beta}}\right)$

$=\sqrt{2}$ [illegible]

III $\sqrt[8]{\alpha\beta^5} + \sqrt[8]{(1-\alpha)(1-\beta)^5} + \sqrt[8]{\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}} = 1$

V $\sqrt[8]{\alpha\beta^3} + \sqrt[8]{(1-\alpha)(1-\beta)^3} + \sqrt[3]{2}\cdot\sqrt[24]{\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}} = 1$

XXXI $\sqrt[32]{\alpha\beta}\Big\{\sqrt[8]{(1+\sqrt{\alpha})(1+\sqrt{\beta})}\sqrt{1+\sqrt[4]{\alpha\beta}+\sqrt{(1-\sqrt{\alpha})(1-\sqrt{\beta})}}$
$+\sqrt[8]{(1-\sqrt{\alpha})(1-\sqrt{\beta})}\sqrt{1+\sqrt[4]{\alpha\beta}+\sqrt{(1+\sqrt{\alpha})(1+\sqrt{\beta})}}\Big\}$
$+\sqrt[32]{(1-\alpha)(1-\beta)}\Big\}$ &c &c &c $\Big\} = \sqrt[4]{8}$

$F\left(\frac{1}{2} - 3\sqrt{5\sqrt{13}-18}\right) = e^{-\pi\sqrt{13}}$

changing β to $\frac{4B}{(1+B)^2}$ & α to $1-B^2$ we get an equation in $\frac{4B(1-B)}{1+B}$ and the value of B^2 is for $e^{-\pi\sqrt{2n}}$

If α I degree, β 3rd & γ 9th

then $1+\sqrt[3]{4}\cdot\sqrt[24]{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}} = 3\sqrt{\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\alpha+\&c}}$

III $\sqrt{\alpha(1-\beta)} + \sqrt{\beta(1-\alpha)} = 2\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$

V $\sqrt[4]{\alpha(1-\beta)} + \sqrt[4]{\beta(1-\alpha)} = \sqrt[3]{4}\sqrt[24]{\alpha\beta(1-\alpha)(1-\beta)}$

Let $\alpha = \sin^2(u+v)$ & $\beta = \sin^2(u-v)$

III $\sin 2u = 2\sin v$; V $\sin 2u = \sin v(1+\cos^2 v)$

VII $\sin 2u = 4\sin\frac{v}{2}\sqrt{\cos 2v + 3\sin^2\frac{v}{2}}$

$$F\;\frac{1-\sqrt{1-(55\pm 12\sqrt{21})(8-3\sqrt{7})^2}}{2}=e^{-\pi\sqrt{21}}$$ ~~160~~ ~~159~~ 147

$$F\;\frac{1-\sqrt{1-\left(\frac{\sqrt{4+\sqrt{7}}-\sqrt[4]{7}}{2}\right)^{24}}}{2}=e^{-7\pi}$$

$$F\;\frac{1\mp\sqrt{1-(\sqrt{5}-2)^8(2-\sqrt{3})^8\left(\frac{\sqrt{4+\sqrt{15}}\pm\sqrt[4]{15}}{2}\right)^{24}}}{2}=e^{-15\pi}$$

If α is 1st deg. β 3rd, γ 5th & δ 15th, then

$$\frac{\sqrt[8]{\beta\gamma}-\sqrt[8]{\beta\gamma(1-\beta)(1-\gamma)}}{\sqrt[4]{\alpha\delta}}=\frac{\sqrt[8]{(1-\beta)(1-\gamma)}-\sqrt[8]{\beta\gamma(1-\beta)(1-\gamma)}}{\sqrt[4]{(1-\alpha)(1-\delta)}}$$

$$=\sqrt{\frac{1+(\frac{1}{4})^2\alpha+\&c}{1+(\frac{1}{4})^2\beta+\&c}\cdot\frac{1+(\frac{1}{4})^2\delta+\&c}{1+(\frac{1}{4})^2\gamma+\&c}}.$$

$$\phi(x)\phi(x^9)=\phi^2(x^3)+2x\,\phi(-x^2)\,\psi(x^9)\,(1+x^3)(1+x^9)(1+x^{15})\,\&c$$

$$\frac{1+(\frac{1}{4})^2\beta+(\frac{1\cdot 3}{2\cdot 4})^2\beta^2+\&c}{\sqrt{1+(\frac{1}{4})^2\alpha+\&c}\,\sqrt{1+(\frac{1}{4})^2\gamma+\&c}}=1-\sqrt[3]{4}\,\sqrt[24]{\frac{\gamma^3(1-\alpha)^3}{\beta(1-\beta)}}=\sqrt[3]{4}\,\sqrt[24]{\frac{\alpha^3(1-\gamma)^3}{\beta(1-\beta)}}-1$$

where α is 1st, β 3rd & γ 9th deg

$$\therefore\ \sqrt[8]{\alpha(1-\gamma)}+\sqrt[8]{\gamma(1-\alpha)}=\sqrt[3]{4}\cdot\sqrt[24]{\beta(1-\beta)}.$$

$$\sqrt{\alpha\gamma}+\sqrt{(1-\alpha)(1-\gamma)}+2\sqrt[3]{4\beta(1-\beta)}$$
$$=1+8\sqrt[4]{\beta(1-\beta)}\,\sqrt[8]{\alpha\gamma(1-\alpha)(1-\gamma)}$$

$$1+\sqrt[3]{4}\,\sqrt[24]{\frac{\gamma^3(1-\gamma)^3}{\beta(1-\beta)}}=\sqrt{\frac{1+(\frac{1}{4})^2\alpha+\&c}{1+(\frac{1}{4})^2\gamma+\&c}}$$

$1-$ [illegible] $\sqrt[24]{\frac{\gamma^3(1-\alpha)^3}{\beta(1-\beta)}}\ \sqrt[24]{\frac{\gamma^3(1-\gamma)^3}{\beta(1-\beta)}}=\frac{1+(\frac{1}{4})^2\beta}{1+(\frac{1}{4})^2\alpha}$ [illegible]

$$\psi(x) - 3x\,\psi(x^9) = \frac{\phi(-x)}{(1-x^3)(1-x^9)(1-x^{15})\,\&c}$$

$$\phi(x)\phi(x^{15}) - \phi(x^3)\phi(x^5)$$
$$= 2x\, f(-x^2) f(-x^{30})\,(1+x^3)(1+x^9)(1+x^{15})\,\&c$$
$$\times (1+x^5)(1+x^{15})(1+x^{25})\,\&c.$$

$$\phi(x)\phi(x^9) + \phi^2(x^3) = 2\psi(x)\phi(-x^{18})(1+x^3)(1+x^9)\,\&c$$

$$\phi(x)\phi(x^{15}) + \phi(x^3)\phi(x^5)$$
$$= 2 f(-x^6) f(-x^{10})\,(1+x)(1+x^3)(1+x^5)\,\&c$$
$$\times (1+x^{15})(1+x^{45})(1+x^{75})\,\&c$$

$$\psi(x^3)\psi(x^5) - x\,\psi(x)\psi(x^{15}) = \frac{f(-x)\,f(-x^{15})}{\chi(-x^3)\,\chi(-x^5)}$$

$$\psi(x^3)\psi(x^5) + x\,\psi(x)\psi(x^{15}) = \frac{f(-x^3)\,f(-x^5)}{\chi(-x)\,\chi(-x^{15})}.$$

$$F\left(\frac{1-\sqrt{1-x^{24}}}{2}\right) = e^{-13\pi}$$

where $x + \frac{1}{x} = \dfrac{\left(\sqrt[3]{3\sqrt{3}+1} + \sqrt[3]{3\sqrt{3}-1}\right)^2}{3\sqrt[3]{2}}\sqrt[6]{13}.$

$$F\left(\frac{1-\sqrt{1-x^{24}}}{2}\right) = e^{-11\pi}$$

where $x + \frac{1}{x} = \dfrac{\sqrt[3]{9\sqrt{3}+1} + \sqrt[3]{9\sqrt{3}-1}}{\sqrt{3}} \cdot \sqrt[6]{\dfrac{11}{2}}.$

$$x^{\frac{3}{8}}\left\{\sqrt{\frac{\sqrt{x}+\frac{1}{\sqrt{x}}+1}{2}} - \sqrt{\frac{\sqrt{x}+\frac{1}{\sqrt{x}}-1}{2}}\right\}$$

$x\left\{\sqrt{A+1} \pm \sqrt{A}\right\}$ where $A = \dfrac{3}{2} \cdot \dfrac{1}{\sqrt{\dfrac{x+\frac{1}{x}+1}{x+\frac{1}{x}-2}} - 1}$

$$\psi^2(-x) + x\psi^2(-x^5) = \frac{f(x^5)\,\phi(x^5)}{\chi(x)}.$$

$$\psi^2(-x) + 5x\psi^2(-x^5) = \frac{\phi^2(x)}{\chi(x)\,\chi(x^5)}$$

III. If $\alpha = \dfrac{1 \pm \sqrt{1-x^3}}{2}$ & $\beta = \dfrac{1-\sqrt{1-y^8}}{2}$

then $y\sqrt{2} = \sqrt[8]{x}\left\{\sqrt{1+x+x^2} - x \;\; \pm \sqrt{(1-x)\left(2\sqrt{1+x+x^2} - 1 - 2x\right)}\right\}.$

$$F\left\{\frac{32 + 9\sqrt{7} - 7\sqrt{3} - (16 - \sqrt{21})\sqrt{2(\sqrt{21} - 3)}}{64}\right\}$$

$$= e^{-3\pi\sqrt{7}}$$

$$F\;\frac{1 - \sqrt{1 - (\sqrt{5}-2)^6(4 \pm \sqrt{15})^4}}{2} = e^{-3\pi\sqrt{5}}$$

$$F\;\frac{1 - \sqrt{1 - \dfrac{4}{\left(\sqrt[3]{1+\sqrt{\frac{11}{27}}} + \sqrt[3]{1-\sqrt{\frac{11}{27}}}\right)^{12}}}}{2} = e^{-\pi\sqrt{11}}$$

$$F\;\frac{1 - \sqrt{1 - \dfrac{4}{\left(\sqrt[3]{1+\sqrt{\frac{23}{27}}} + \sqrt[3]{1-\sqrt{\frac{23}{27}}}\right)^{24}}}}{2} = e^{-\pi\sqrt{23}}$$

$$F\;\frac{1 - \sqrt{1 - \dfrac{2^6}{\left(\sqrt[3]{1+\sqrt{\frac{19}{27}}} + \sqrt[3]{1-\sqrt{\frac{19}{27}}}\right)^{24}}}}{2}$$

$$F\,\frac{1-\sqrt{1-\left(\frac{\sqrt[3]{1+\sqrt{\frac{31}{27}}}+\sqrt[3]{1-\sqrt{\frac{31}{27}}}}{2^{4}}\right)^{24}}}{2}=e^{-\pi\sqrt{31}}$$

XVII $$\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}+\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}-2\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left\{1+\sqrt[8]{\frac{\beta}{\alpha}}+\sqrt[8]{\frac{1-\beta}{1-\alpha}}\right\}=\frac{1+\left(\frac{1}{2}\right)^2\alpha+\cdots}{1+\left(\frac{1}{2}\right)^2\beta+\cdots}$$

$$F\,\frac{1-\sqrt{1-\left(\frac{1}{\sqrt{5-\sqrt{17}}}-\frac{1}{\sqrt{3+\sqrt{17}}}\right)^{24}}}{2}=e^{-\pi\sqrt{17}}$$

XXXIII ? $$\sqrt{\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}+\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}-2\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left\{1+\sqrt[8]{\frac{\beta}{\alpha}}+\sqrt[8]{\frac{1-\beta}{1-\alpha}}\right\}-2\sqrt[16]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\sqrt[4]{\sqrt[8]{\frac{\beta}{\alpha}}+\sqrt[8]{\frac{1-\beta}{1-\alpha}}+\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}}}$$
$$=\sqrt{\frac{1+\left(\frac{1}{2}\right)^2\alpha+\left(\frac{1\cdot3}{2\cdot4}\right)^2\alpha^2+\cdots}{1+\left(\frac{1}{2}\right)^2\beta+\left(\frac{1\cdot3}{2\cdot4}\right)^2\beta^2+\cdots}}$$

IX $$\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{1-\beta}{1-\alpha}}+\sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}-4\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left\{1+\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}\right\}=\left\{\frac{1+\left(\frac{1}{2}\right)^2\alpha+\cdots}{1+\left(\frac{1}{2}\right)^2\beta+\cdots}\right\}$$

VII $$\left\{\sqrt[4]{\frac{(1-\beta)^7}{1-\alpha}}+\sqrt[4]{\frac{\beta^7}{\alpha}}+1\right\}-2\left\{\sqrt[8]{\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}}+\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}}+\sqrt[8]{[illegible]}\right\}$$
$$-4\sqrt[4]{\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}}\left\{\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}}+\sqrt[8]{\frac{\beta^7}{\alpha}}+1\right\}-3\sqrt[12]{\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}}=0$$

~~[illegible] part of~~ $2x\,\psi\left(x^{\frac{8m}{m+n}}\right)\psi\left(x^{\frac{8n}{m+n}}\right)$

~~[illegible] function~~ $=\phi(x)\,\phi$ [illegible]

[illegible] complementary [illegible]

III $\sqrt[8]{\alpha\beta^5} + \sqrt[8]{(1-\alpha)(1-\beta)^5} = \sqrt{1-\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}}$

V $\sqrt[8]{\alpha\beta^3} + \sqrt[8]{(1-\alpha)(1-\beta)^3} = \sqrt{1-\sqrt[6]{16\alpha\beta(1-\alpha)(1-\beta)}}$ [illegible]

149

III $$\frac{1+\sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}}}{\sqrt{1-\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}}} = \frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}$$

VII $$\frac{\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}} - \sqrt[8]{\frac{\beta^7}{\alpha}}}{1-\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}} = \frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}$$

$$\frac{\phi^2(e^{-7\pi})}{\phi^2(e^{-\pi})} = \left(\frac{\sqrt{13+\sqrt{7}}+\sqrt{7+3\sqrt{7}}}{\cancel{\sqrt[8]{7}}\ 14}\right)\sqrt[8]{28}$$

$$\phi(x^3)\phi(x^5) = \phi(-x^2)\phi(-x^{30}) + 2x^2\psi(x)\psi(x^{15}).$$

If I α, 3rd β, 5th γ & 15th δ then

$$\sqrt[8]{\alpha\delta} + \sqrt[8]{(1-\alpha)(1-\delta)} = \sqrt{\frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\alpha+\&c}\cdot\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\delta+\&c}}$$

$$\sqrt[8]{\beta\gamma} + \sqrt[8]{(1-\beta)(1-\gamma)} = \sqrt{\frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}\cdot\frac{1+(\frac{1}{2})^2\delta+\&c}{1+(\frac{1}{2})^2\gamma+\&c}}$$

$$\sqrt[8]{\alpha\delta} - \sqrt[8]{(1-\alpha)(1-\delta)} = \sqrt[8]{\beta\gamma} - \sqrt[8]{(1-\beta)(1-\gamma)}$$

$$\sqrt[8]{\beta\gamma} + \sqrt[8]{(1-\beta)(1-\gamma)} = \sqrt[3]{4}\cdot\sqrt[24]{\frac{\beta^2\gamma^2(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)(1-\delta)}} - 1.$$

$$\sqrt[8]{\alpha\delta} + \sqrt[8]{(1-\alpha)(1-\delta)} = -\sqrt[3]{4}\cdot\sqrt[24]{\frac{\alpha^2\delta^2(1-\alpha)^2(1-\delta)^2}{\beta\gamma(1-\beta)(1-\gamma)}} + 1.$$

VII $$\frac{\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}} - 1}{\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}} - \sqrt[8]{\frac{\beta^7}{\alpha}}} = \sqrt[8]{\alpha\beta}.$$

1st α, 5th β, 7th γ & 35th δ

$$\sqrt[4]{\alpha\delta} + \sqrt[4]{(1-\alpha)(1-\delta)} + 2\sqrt[3]{2}\sqrt[12]{\alpha\delta(1-\alpha)(1-\delta)}$$

$$+ \sqrt[4]{\beta\gamma} + \sqrt[4]{(1-\beta)(1-\gamma)} + 2\sqrt[3]{2}\sqrt[12]{\beta\gamma(1-\beta)(1-\gamma)}$$

$$= 1 + \left\{1 + 2\sqrt[3]{2}\sqrt[24]{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}\right\}^2.$$

$$P = \sqrt[8]{\frac{\beta}{\alpha}} + \sqrt[8]{\frac{1-\beta}{1-\alpha}} + \sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$$

$$Q = \sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left\{\sqrt[8]{\frac{\beta}{\alpha}} + \sqrt[8]{\frac{1-\beta}{1-\alpha}} + 1\right\}$$

$$R = \sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$$

17. $\sqrt{P^2 - 4Q} = K.$

49. $P + R^{\frac{1}{3}} - \sqrt{4Q + 4PR^{\frac{1}{3}} + 13R^{\frac{2}{3}}} = K.$

V
$$\frac{m + 2(m-n)\sqrt[3]{2}\sqrt[24]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}} + n\sqrt[3]{4}\sqrt[12]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}}}{m - n\sqrt[6]{16\alpha\beta(1-\alpha)(1-\beta)}}$$

$$= \frac{1 - \sqrt[3]{2}\sqrt[24]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}} - \sqrt[3]{4}\sqrt[12]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}}}{\sqrt{1 - 3\sqrt[6]{16\alpha\beta(1-\alpha)(1-\beta)} + \sqrt[3]{16\alpha\beta(1-\alpha)(1-\beta)}}} = \frac{1 + (\tfrac{1}{2})^2\alpha + \ldots}{1 + (\tfrac{1}{2})^2\beta + \ldots}$$

XI
$$\frac{1 + 8\sqrt[3]{2}\sqrt[24]{\frac{\beta^{11}(1-\beta)^{11}}{\alpha(1-\alpha)}}}{\left\{\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right\}\left\{\sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)} + \sqrt{2 + 2\sqrt{\alpha\beta} + 2\sqrt{(1-\alpha)(1-\beta)}}\right\}}$$

$$= \frac{1 + (\tfrac{1}{2})^2\alpha + (\tfrac{1\cdot 3}{2\cdot 4})^2\alpha^2 + \ldots}{1 + (\tfrac{1}{2})^2\beta + (\tfrac{1\cdot 3}{2\cdot 4})^2\beta^2 + \ldots}$$

VII
$$\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}} + \sqrt[8]{\frac{\beta^7}{\alpha}} + 2\sqrt[24]{\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}}$$

$$= [illegible] \left\{\frac{1 + (\tfrac{1}{2})^2\alpha + \ldots}{1 + (\tfrac{1}{2})^2\beta + \ldots}\right\}^2.$$

1st α, 5th β, 7th γ & 35th δ

$$\left\{\sqrt[4]{\alpha\delta}+\sqrt[4]{(1-\alpha)(1-\delta)}+2\sqrt[3]{2}\sqrt[12]{\alpha\delta(1-\alpha)(1-\delta)}\right\}$$
$$\times\left\{\sqrt[4]{\beta\gamma}+\sqrt[4]{(1-\beta)(1-\gamma)}+2\sqrt[3]{2}\sqrt[12]{\beta\gamma(1-\beta)(1-\gamma)}\right\}$$
$$=1-4\sqrt[3]{2}\sqrt[24]{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}\left\{\sqrt[8]{\alpha\beta\gamma\delta}+\sqrt[8]{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}\right\}$$

$$\phi(x)\phi(x^{27})-\phi(-x)\phi(-x^{27})=4xf(-x^6)f(-x^{18})+4x^7\psi(x^2)\psi(x^{54})$$

$$\phi(x)\phi(x^{35})-\phi(-x)\phi(-x^{35})=4xf(-x^{10})f(-x^{14})+4x^9\psi(x^2)\psi(x^{70})$$

$$\phi(x^5)\phi(x^7)-\phi(-x^5)\phi(-x^7)=4x^3\psi(x^{10})\psi(x^{14})-4x^{9}f(-x^2)f(-x^{70})$$

$$\phi(x)\phi(x^{63})-\phi(x^7)\phi(x^9)=2xf(x^3)f(x^{21})$$

If α, β, γ, δ = 1, 3, 13, 39 or 1, 5, 11, 55 or 1, 7, 9, 63

then $$\frac{1+\sqrt[4]{(1-\alpha)(1-\delta)}+\sqrt[4]{\alpha\delta}}{1+\sqrt[4]{(1-\beta)(1-\gamma)}+\sqrt[4]{\beta\gamma}}=\frac{\sqrt[8]{(1-\alpha)(1-\delta)}-\sqrt[8]{\alpha\delta}}{\sqrt[8]{(1-\beta)(1-\gamma)}-\sqrt[8]{\beta\gamma}}$$
$$=\sqrt{\frac{1+(\frac{1}{4})^2\beta+\&c}{1+(\frac{1}{4})^2\alpha+\&c}\cdot\frac{1+(\frac{1}{4})^2\gamma+\&c}{1+(\frac{1}{4})^2\delta+\&c}}$$
$$=\frac{\sqrt[8]{\alpha\delta}\pm\sqrt[8]{\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[8]{\beta\gamma}-\sqrt[8]{\beta\gamma(1-\beta)(1-\gamma)}}\quad\left(\begin{matrix}+\text{ in the I 2 cases \&}\\-\text{ in 1, 7, 9, 63}\end{matrix}\right)$$

$$\psi(x^7)\psi(x^9)-x^6\psi(x)\psi(x^{63})=f(-x^6)f(-x^{42}).$$

$$\phi(x)\phi(x^{135})=\phi(-x^{10})\phi(-x^{54})+2x^4\psi(x^5)\psi(x^{27})+2xf(x^9)f(\ldots)$$

$$\phi(x^5)\phi(x^{27})=\phi(-x^2)\phi(-x^{270})+2x^{17}\psi(x)\psi(x^{135})+2x^2f(x^3)f(x^{45})$$

$$\text{III}\quad M^2\sqrt{\alpha(1-\alpha)} + \sqrt{\beta(1-\beta)} = 2\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$$

$$\text{V}\quad M\sqrt[4]{\alpha(1-\alpha)} + \sqrt[4]{\beta(1-\beta)} = 3/4\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}$$

$$\text{III}\quad \begin{cases} M\sqrt{1-\alpha} + \sqrt{1-\beta} = 2\sqrt[8]{(1-\alpha)(1-\beta)} \\ M\sqrt{\alpha} - \sqrt{\beta} = 2\sqrt[8]{\alpha\beta} \end{cases}$$

$$\left.\begin{aligned} &\text{II}\quad M\sqrt{1-\alpha} + \sqrt{\beta} = 1 \\ &\text{IV}\quad \sqrt{M}\sqrt[4]{1-\alpha} + \sqrt[4]{\beta} = 1 \end{aligned}\right\} \qquad \left.\begin{aligned} &\text{II}\quad M^2\sqrt{1-\alpha} + \beta = 1 \\ &\text{IV}\quad M\sqrt[4]{1-\alpha} + \sqrt{\beta} = 1 \\ &\text{VIII}\quad \sqrt{M}\sqrt[8]{1-\alpha} + \sqrt[4]{\beta} = 1 \end{aligned}\right\}$$

$$\text{II}\quad M^2 = 2\cdot\frac{1+\beta}{1+(1-\alpha)} = 2\cdot\frac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}}$$

$$\text{IV}\quad M = 2\cdot\frac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}} = 2\cdot\frac{1+\sqrt[4]{\beta}}{1+\sqrt[4]{1-\alpha}}.$$

$$\text{XVI}\quad \sqrt{M} = 2\cdot\frac{1+\sqrt[4]{\beta}}{1+\sqrt[4]{1-\alpha}}.$$

$$\text{IV.}\quad \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{1-\beta}{1-\alpha}} - \frac{4}{M}\sqrt[3]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} = M.$$

$$\text{VII}\quad \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{1-\beta}{1-\alpha}} - \frac{7}{M}\sqrt[3]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 3\sqrt[6]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} = M.$$

$$\text{IX}\quad M = 3\cdot\frac{1+2\sqrt[3]{\beta}}{1+2\sqrt[3]{1-\alpha}}.$$

Equations for $1+\frac{1.3}{4^2}x+\frac{1.3.5.7}{4^2.8^2}x^2+\&c$

(3) $\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{1-\beta}{1-\alpha}}-\frac{9}{M^2}\sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}=M^2$

(5) $\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}-\frac{5}{M}\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}=M$

(9) $\sqrt[8]{\frac{\beta}{\alpha}}+\sqrt[8]{\frac{1-\beta}{1-\alpha}}-\frac{3}{\sqrt{M}}\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}=\sqrt{M}.$

(7) $\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{1-\beta}{1-\alpha}}-\frac{49}{M^2}\sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}-8\sqrt[6]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left(\sqrt[6]{\frac{\beta}{\alpha}}+\sqrt[6]{\frac{1-\beta}{1-\alpha}}\right)$

$=M^2$

(13) $\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}-\frac{13}{M}\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}-4\sqrt[12]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left(\sqrt[12]{\frac{\beta}{\alpha}}+\sqrt[12]{\frac{1-\beta}{1-\alpha}}\right)=M$

(25) $\sqrt[8]{\frac{\beta}{\alpha}}+\sqrt[8]{\frac{1-\beta}{1-\alpha}}-\frac{5}{\sqrt{M}}\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}-2\sqrt[24]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left(\sqrt[24]{\frac{\beta}{\alpha}}+\sqrt[24]{\frac{1-\beta}{1-\alpha}}\right)=\sqrt{M}$

(2). $1+\frac{1.3}{4^2}\left\{1-\left(\frac{1-t}{1+3t}\right)^2\right\}+\&c$

$=\sqrt{1+3t}\left\{1+\frac{1.3}{4^2}t^2+\frac{1.3.5.7}{4^2.8^2}t^4+\right\}$

(3) If $\alpha=\frac{64t}{(9+6t-t^2)^2}$ $\beta=\frac{64t^3}{(27-18t-t^2)^2}$

then $\sqrt{1-\frac{2}{3}t-\frac{t^2}{27}}\left\{1+\frac{1.3}{4^2}\alpha+\right\}$.

$=\sqrt{1+2t-\frac{t^2}{3}}\left\{1+\frac{1.3}{4^2}\beta+\right\}$.

$1+12\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)=\phi^2(x)\phi^2(x^3)$. $\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)\ldots}}{2}$

$1+4\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)=\phi^2(x)\phi^2(x^7)$. $\frac{1+\sqrt{\alpha\beta}+\sqrt{\ldots}}{2}$

$$P = 1 - \sqrt[8]{\alpha\beta} - \sqrt[8]{(1-\alpha)(1-\beta)}$$

$$Q = 4\left\{\sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)} - \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\right\}$$

$$R = 4\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}.$$

7. $P = 0$. 23. $P - R^{\frac{1}{3}} = 0$.

39. ~~$P^5 - R(6P^2+Q) + \frac{R^2}{P} + \frac{R^2P^2}{P^3-R} = 0$.~~

55. ~~$P^7 - R(18P^4 + 9P^2Q + Q^2)$~~ $+ 10R^2P + \frac{QR^2}{P}$

$$+ \frac{R^2P^2Q}{P^3-R} = 0.$$

71. $P^3 - R^{\frac{1}{3}}(4P^2+Q) + 2PR^{\frac{2}{3}} - R = 0$.

~~87. $P^{11} -$~~

15. $P^2 - Q + \frac{R}{P} = 0$.

31. $P^2 - Q - \sqrt{PR} = 0$

47. $P^2 - Q - PR^{\frac{1}{3}} - 2R^{\frac{2}{3}} = 0$.

~~63~~. 95. $(P^2-Q)^2 - PR^{\frac{1}{3}}(5P^2 - 4Q) + QR^{\frac{2}{3}}$

$$- R\left(P + \frac{Q}{P}\right) - 5R^{\frac{4}{3}} = 0$$

3. $P = 0$ 11. $P - R^{\frac{1}{3}} = 0$. 19. $P^5 - 7P^2R - QR = 0$

35. $P^3 - R^{\frac{1}{3}}(5P^2+Q) + 2R^{\frac{2}{3}}P - R - \frac{R^{\frac{4}{3}}}{P} = 0$.

~~51. P^5~~ 27. $P^7 - R(29P^4 + 11P^2Q + Q^2) - 17R^2P$

$$- 3\frac{R^2}{P^2}(PQ + R)$$

$= 0$

7, 23 &c

$$\begin{cases} P = x f(x^2, x^{14}) - x^{\frac{n+1}{8}} \psi(x^2) \\ Q = \{\phi(-x^2) + 4 x^{\frac{n+9}{8}} f(x^2, x^{14})\} \psi(x^2). \\ R = x^{\frac{n+1}{8}} f^3(-x^4). \end{cases}$$

3, 11 &c

$$\begin{cases} P = x \psi(x^8) - x^{\frac{n+1}{4}} \psi(x^4) \\ Q = \phi^2(-x^2) + 16 x^{\frac{n+5}{4}} \psi(x^2) \psi(x^8). \\ R = x^{\frac{n+1}{4}} f^3(-x^2). \end{cases}$$

1, 5 &c

$$\begin{cases} P = \{x \psi(x^4) - 2 x^{\frac{n+1}{2}} \phi(x^2)\} \psi(x^4). \\ Q = \phi^4(-x^2) + 128 x^{\frac{n+3}{2}} \psi^2(x^2) \psi^2(x^4). \\ R = x^{\frac{n+1}{2}} f^6(-x^2). \end{cases}$$

15, 31 &c.

$$\begin{cases} P = f(x^6, x^{10}) + x^{\frac{n+1}{8}} \psi(x^2). \\ Q = \{\phi(-x^2) + 4 x^{\frac{n+1}{8}} f(x^6, x^{10})\} \psi(x \ldots \\ R = x^{\frac{n+1}{8}} f^3(-x^4). \\ P_2 = P^2 - Q = x^2 f^2(x^2, x^{14}) - 2x^{\frac{n+1}{8}} f(x^6, x \ldots \\ \qquad \times \psi(x^2) + x^{\frac{n+1}{4}} \psi^2(x^2). \end{cases}$$

$P = 1 + \sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)}$	$1 - \sqrt[4]{\alpha\beta} - \sqrt[8]{(1-\alpha)(1-\ldots}$
$Q = 4\{\sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)} + \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\}$	16.
$R = 4 \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$	16.
$P = 1 - \sqrt[8]{\alpha\beta} - \sqrt[8]{(1-\alpha)(1-\beta)}$	$1 - \sqrt{\alpha\beta} - \sqrt{(1-\ldots}$
$Q = 4.$	64.
$R = 4.$	32.

. $P - R^{\frac{1}{3}} = 0.$ 9. $P^5 - R(14P^2 + Q) - \frac{3R^2}{P} = 0.$

13. $\sqrt{P}(P^3 + 8R) - \sqrt{R}(11P^2 + Q) = 0.$

17. $P^3 - R^{\frac{1}{3}}(10P^2 + Q) + 18R^{\frac{2}{3}}P + 12R = 0.$

29. $\sqrt{P}(P^2 + 17R^{\frac{1}{3}}P - 9R^{\frac{2}{3}})$

$- \sqrt[6]{R}(9P^2 + Q - 13R^{\frac{1}{3}}P + 15R^{\frac{2}{3}}) = 0.$

9. $P^5 - R^{\frac{1}{3}}(14P^4 + 9P^2Q + Q^2) + R^{\frac{2}{3}}P(19P^2 + Q)$

$+ 6R(7P^2 + Q) + 4R^{\frac{4}{3}}P - 3R^{\frac{5}{3}} = 0.$

21.

$1 + 6\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \frac{3x^6}{1-x^6} + \&c\right)$ 153 ~~166~~

$= \phi^2(x)\,\phi^2(x^5)$. $\frac{3+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{4}$ $\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$

$$F\left(\frac{1-\sqrt{1-\frac{1}{4}(\frac{3}{2}\sqrt{2}-1)^8}}{2}\right) = e^{-3\pi\sqrt{3}}$$

$$F\left(\frac{1-\sqrt{1-(\sqrt{37}-6)^6}}{2}\right) = e^{-\pi\sqrt{37}}$$

$$F\left(\frac{1-\sqrt{1-4\cdot\left(\frac{\sqrt{13}-3}{8}\right)^4\left(\sqrt{\frac{5+\sqrt{13}}{8}} \pm \sqrt{\frac{\sqrt{13}-3}{8}}\right)^{24}}}{2}\right) = e^{-\pi\sqrt{39}}$$

$$F\left(\frac{1-\sqrt{1-\left(\sqrt{\frac{13+\sqrt{97}}{8}} - \sqrt{\frac{5+\sqrt{97}}{8}}\right)^{24}}}{2}\right) = e^{-\pi\sqrt{97}}$$

$$F\left(\frac{1-\sqrt{1-\left(\sqrt{\frac{15+\sqrt{193}}{4}} - \sqrt{\frac{11+\sqrt{193}}{4}}\right)^{24}}}{2}\right) = e^{-\pi\sqrt{4}}$$

$$F\left(\frac{1-\sqrt{1-\frac{1}{4}\cdot\left(\frac{5-\sqrt{21}}{4}\right)^4\left\{\sqrt{\frac{5+\sqrt{21}}{8}} \pm \sqrt{\frac{\sqrt{21}-3}{8}}\right\}^{24}}}{2}\right) = e^{-\pi\sqrt{63}}$$

$$\phi(-x)\phi(-x^{95}) + 4x f(x^5) f(x^{19})$$
$$= \phi(x)\phi(x^{95}) + 4x^6 f(-x^3, -x^5) f(-x^{285}, -x^{475})$$

$$\phi(-x)\phi(-x^{63}) + 4x f(x^3) f(x^{21})$$
$$= \phi(x)\phi(x^{63}) + 4x^4 f(-x^3, -x^5) f(-x^{189}, -x^{315}).$$

$$\phi(-x)\phi(-x^{143}) + 4x f(x^{11}) f(x^{13})$$
$$= \phi(x)\phi(x^{143}) - 4x^9 f(-x^3, -x^5) f(-x^{429}, -x^{715}).$$

7. $$\sqrt[4]{\frac{(1-\beta)^3}{1-\alpha}} + \sqrt[4]{\frac{\beta^3}{\alpha}} - \sqrt[4]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}} = \frac{m^2}{2}\left\{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}\right\}$$

1. $$\sqrt[8]{\frac{(1-\beta)^3}{1-\alpha}} - \sqrt[8]{\frac{\beta^3}{\alpha}} - \sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}} ~~= \frac{m}{2}\left\{1 + \sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right\}~~$$
$$= m\sqrt{\frac{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}}{2}}$$

$$m - \frac{3}{m} = 2\left(\sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)}\right)$$

$$m - \frac{5}{m} = 4\left(\sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)}\right) \Big/ \sqrt{\frac{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}}{2}}$$

$$m - \frac{7}{m} = 2\left(\sqrt[8]{\alpha\beta} - \sqrt[8]{(1-\alpha)(1-\beta)}\right)\left(2 + \sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right)$$

$$m - \frac{11}{m} = 2\left(\sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)}\right)\left(4 + \sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right)$$

7:— $\cos 2u = (2\cos v - 1)\sqrt{4\cos v - 3}$

$$\sqrt{\frac{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}}{2}} = 1 - \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$$

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1, 3, 5, 15

$$\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}}+\sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}}-\sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}=-\sqrt{\frac{1+(\frac{1}{2})^2\alpha+}{1+(\frac{1}{2})^2\beta}\cdot\frac{1+(\frac{1}{2})}{1+(\frac{1}{2})}}$$

$$\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}}+\sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}}-\sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}=\sqrt{\frac{1+(\frac{1}{2})^2\beta}{1+(\frac{1}{2})^2\alpha}\cdot\frac{1+(\frac{1}{2})^2\gamma}{1+(\frac{1}{2})^2\delta}}$$

$$\sqrt[8]{\alpha\beta\gamma\delta}+\sqrt[8]{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}$$
$$+\sqrt[3]{2}\cdot\sqrt[24]{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}=1$$

1, 5, 25

$$\frac{1+\sqrt[3]{4}\cdot\sqrt[24]{\dfrac{\beta^{10}(1-\beta)^{10}}{\alpha\gamma(1-\alpha)(1-\gamma)}}}{1+\sqrt[3]{4}\cdot\sqrt[24]{\dfrac{\alpha^5\gamma^5(1-\alpha)^5(1-\gamma)^5}{\beta^2(1-\beta)^2}}}=\frac{\phi^2(x)\,\phi^2(x^{25})}{\phi^4(x^5)},$$

1, 3, 13, 39 or 1, 5, 7, 35.

$$\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}}+\sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}}-\sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}+2\sqrt[12]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$

+ in 13 & − in 7.

$$=\pm\sqrt{\frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}\cdot\frac{1+(\frac{1}{2})^2\delta+\&c}{1+(\frac{1}{2})^2\gamma+\&c}}$$

$$\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}}+\sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}}-\sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}+2\sqrt[12]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}$$

$$=\sqrt{\frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\alpha+\&c}\cdot\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\delta+\&c}}$$

1, 3, 9

$$\sqrt[4]{\frac{\beta^2}{\alpha\gamma}}+\sqrt[4]{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}}-\sqrt[4]{\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}}=-3\cdot\frac{1+(\frac{1}{2})^2\alpha}{1+(\frac{1}{2})^2\beta}\cdot\frac{1+(\frac{1}{2})^2\gamma+}{1+(\frac{1}{2})^2\beta+}$$

$$\sqrt[4]{\frac{\alpha\gamma}{\beta^2}}+\sqrt[4]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}}-\sqrt[4]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}}=\frac{1+(\frac{1}{2})^2\beta+}{1+(\frac{1}{2})^2\alpha+}\cdot\frac{1+(\frac{1}{2})^2\beta+}{1+}$$

154

125

1, 3, 7, 21.

$$\sqrt[4]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[4]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[4]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} + 4\sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$

$$= \sqrt{\frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}\cdot\frac{1+(\frac{1}{2})^2\delta+\&c}{1+(\frac{1}{2})^2\gamma+\&c}}$$

$$\sqrt[4]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[4]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} - \sqrt[4]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} + 4\sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}$$

$$= \sqrt{\frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\alpha+\&c}\cdot\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\delta+\&c}}$$

1, 3, 9, 27

$$\left[\sqrt[4]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[4]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} + \sqrt[4]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} - 2\sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}\left\{1+\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}}+\sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}}\right\}\right] = -3\cdot\frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}\times\frac{1+(\frac{1}{2})^2\delta+\&c}{1+(\frac{1}{2})^2\gamma+\&c}$$

$$\left[\sqrt[4]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[4]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} + \sqrt[4]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} - 2\sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}\left\{1+\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}}+\sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}}\right\}\right] = \frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\alpha+\&c}\times\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\delta+\&c}$$

1, 5, 25

$$\sqrt[4]{\frac{\beta^2}{\alpha\gamma}} + \sqrt[4]{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}} + \sqrt[4]{\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}} - 2\sqrt[8]{\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}}\times$$

$$\left\{1+\sqrt[8]{\frac{\beta^2}{\alpha\gamma}}+\sqrt[8]{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}}\right\} = 5\cdot\frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}\cdot\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\beta+\&c}$$

$$\sqrt[4]{\frac{\alpha\gamma}{\beta^2}} + \sqrt[4]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}} + \sqrt[4]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}} - 2\sqrt[8]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}}$$

$$\times\left\{1+\sqrt[8]{\frac{\alpha\gamma}{\beta^2}}+\sqrt[8]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}}\right\} =$$

1, 5, 25

$$\sqrt[8]{\frac{\alpha\gamma}{\beta^2}} + \sqrt[8]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}} + \sqrt[8]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}}$$

$$= \frac{\left\{1+\left(\tfrac{1}{2}\right)^2\beta+\right\}^2}{\sqrt{\left\{1+\left(\tfrac{1}{2}\right)^2\alpha+\&c\right\}\left\{1+\left(\tfrac{1}{2}\right)^2\gamma+\right\}}}$$

1, 3, 9, 27

$$\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} + \sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}$$

$$= \sqrt{\frac{1+\left(\tfrac{1}{2}\right)^2\beta+}{1+\left(\tfrac{1}{2}\right)^2\alpha+} \cdot \frac{1+\left(\tfrac{1}{2}\right)^2\gamma+}{1+\left(\tfrac{1}{2}\right)^2\delta+}}$$

N.B. For 1, 5, 7, 35 Same as for 1, 3, 11, 33 $4R^{\frac{2}{3}}$ instead of $2R^{\frac{2}{3}}$

For 1, 3, 9, 27 :- $P^5 - R(11P^2+Q) + 9\frac{R^2}{P} + 6\frac{R^3}{P^4} = 0.$

1, 7, 21, 3 or 1, 3, 33, 11.

$$\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} - 2\sqrt[12]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$

$$= \sqrt{\frac{1+\left(\tfrac{1}{2}\right)^2\alpha}{1+\left(\tfrac{1}{2}\right)^2\beta} \cdot \frac{1+\left(\tfrac{1}{2}\right)^2\delta}{1+\left(\tfrac{1}{2}\right)^2\gamma}}$$

1, 3, 15, 5

$$\sqrt[4]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[4]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[4]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} - 4\sqrt[6]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$

$$= \frac{1+\left(\tfrac{1}{2}\right)^2\alpha}{1+\left(\tfrac{1}{2}\right)^2\beta} \cdot \frac{1+\left(\tfrac{1}{2}\right)^2\delta}{1+\left(\tfrac{1}{2}\right)^2\gamma}$$

For 1, 3, 7, 21 the same eqn. 1, 3, 3, 9; $P^2 - Q + \frac{3R}{P}$ =

1, 5, 5, 25; $P^3 - R^{\frac{1}{3}}(5P^2+Q) - R^{\frac{2}{3}}P + 3R - \frac{R^{\frac{4}{3}}}{P} = 0$

1, 3, 17, 51; $P^3 - R^{\frac{1}{3}}(7P^2+Q) + 13R^{\frac{2}{3}}P - 12R = 0.$

1, 5, 11, 55; $P^3 - R^{\frac{1}{3}}(4P^2+Q) - R^{\frac{2}{3}}P + 4R = 0.$

37. $P^3 - R^{\frac{1}{3}}(7P^2 + Q) - 3PR^{\frac{2}{3}} - 25R - M(19P^2 - \&c) + \&c$

1, 4, 5, 20 or 1, 2, 7, 14

$$\frac{1 + 2\left\{\sqrt[3]{\alpha\delta} + \sqrt[3]{(1-\alpha)(1-\delta)}\right\}}{1 + 2\left\{\sqrt[3]{\beta\gamma} + \sqrt[3]{(1-\beta)(1-\gamma)}\right\}} = \frac{1 + \frac{1.2}{5^2}\beta + \&}{1 + \frac{1.2}{5^2}\alpha + \&} \cdot \frac{1 + \frac{1.2}{5^2}\gamma +}{1 + \frac{1.2}{5^2}\delta +}$$

1, 2, 4, 8 — $1 - \sqrt[3]{\alpha\delta} - \sqrt[3]{(1-\alpha)(1-\delta)}$

$$= 3\sqrt[6]{\beta\gamma(1-\beta)(1-\gamma)} \cdot \frac{1 + \frac{1.2}{5^2}\beta + \&}{1 + \frac{1.2}{5^2}\alpha +} \cdot \frac{1 + \frac{1.2}{5^2}\gamma +}{1 + \frac{1.2}{5^2}\delta +}$$

$\sqrt{2}$. $F(\sqrt{2}-1)^2 = e^{-\pi\sqrt{2}}$

$\sqrt{6}$. $(2-\sqrt{3})^2(\sqrt{3}\pm\sqrt{2})^2$

$\sqrt{10}$ $(\sqrt{10}-3)^2(3\pm\sqrt{8})^2$

$\sqrt{18}$ $(5\sqrt{2}-7)^2(7\pm4\sqrt{3})^2$

$\sqrt{22}$ ~~$(\sqrt{66}-\sqrt{65})^2(\sqrt{65}-\sqrt{64})^2$~~ $(10-3\sqrt{11})^2(3\sqrt{11}\pm7\sqrt{2})^2$

$\sqrt{70}$ $(6-\sqrt{35})^2(15-4\sqrt{14})^2(8-3\sqrt{7})^2(3\sqrt{14}-5\sqrt{5})^2$

$\sqrt{58}$ $(13\sqrt{58}-99)^2(99\pm70\sqrt{2})^2$.

$\sqrt{30}$ $(2-\sqrt{3})^2(5-2\sqrt{6})^2(4-\sqrt{15})^2(\sqrt{6}-\sqrt{5})^2$

Let $\alpha\beta = 16t(1-t)^3$ & $(1-\alpha)(1-\beta) = 16t^3(1-t)$

$\frac{p(1-p)}{1+2p} = 2t$. then $F\left(p^3 \cdot \frac{2+p}{1+2p}\right) = e^{-\pi\sqrt{3n}}$

$4t(1-t) = k$

$$\frac{1 - \sqrt{1-4k}}{\quad}\left(1 - 2k \pm \sqrt{1-k \cdot 1-4k}\right)^2 = e^{-\pi\sqrt{3n}}$$

$$F\,\frac{1-\sqrt{1-(10\pm 3\sqrt{11})^2(2-\sqrt{3})^6}}{2} = e^{-\pi\sqrt{33}}.$$

(−1)

$$F\,\frac{1-\sqrt{1-\dfrac{\left(\frac{\sqrt{5}+1}{2}\sqrt[3]{10}+\frac{\sqrt{5}-1}{2}\sqrt[3]{4\sqrt{5}}-\sqrt{5}-1\right)^{24}}{3^{24}\cdot 2^{10}}}}{2}$$

$$= e^{-5\pi\sqrt{3}}.$$

$$F\,\frac{1+\sqrt{1-\dfrac{\left(\frac{\sqrt{5}+1}{2}\sqrt[3]{4\sqrt{5}}-\frac{\sqrt{5}-1}{2}\sqrt[3]{10}+\sqrt{5}-1\right)^{24}}{3^{24}\cdot 2^{10}}}}{2}$$

$$= e^{-\frac{\pi}{5}\sqrt{3}}.$$

$$\sqrt[8]{1\pm\sqrt{1-(\sqrt{5}-2)^8}} = \frac{\sqrt{5}-1}{2}\cdot\frac{\sqrt[4]{5}\pm 1}{\sqrt{2}}$$

$$\sqrt[8]{1\pm\sqrt{1-(2-\sqrt{3})^4}} = \frac{\sqrt{2}\cdot\sqrt[4]{3}\pm(\sqrt{3}-1)}{2\sqrt[4]{2}}.$$

If $a=\sqrt[4]{60}$, $b = 2-\sqrt{3}+\sqrt{5}$ & $2c = 1+\frac{a+b}{a-b}\sqrt{5}$

then $(\sqrt{c^2+1}-c)\sqrt[5]{e^{6\pi}} = \frac{1}{1+}\,\frac{e^{-6\pi}}{1+}\,\frac{e^{-12\pi}}{1+}$ &c

If $2c = 1+\frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}\sqrt{5}$ then $\frac{e^{-\frac{4\pi}{5}}}{1+}\,\frac{e^{-4\pi}}{1+}\,\frac{e^{-8\pi}}{1+}$ &c

If $a = 3+\sqrt{2}-\sqrt{5}$ & $b=\sqrt[4]{20}$. then $\frac{e^{-\frac{8\pi}{5}}}{1+}\,\frac{e^{-8\pi}}{1+}\,\frac{e^{-16\pi}}{1+}$ &c

If $a = \sqrt[4]{5}(2-\sqrt{2})$ & $b = 1+\sqrt{2}+\sqrt{5}$
$\qquad - \sqrt[4]{2}(3-\sqrt{2}+\sqrt{5}-\sqrt{10})$

then $\frac{e^{-\frac{16\pi}{5}}}{1+}\,\frac{e^{-16\pi}}{1+}\,\frac{e^{-32\pi}}{1+}$ &c

$$\phi(e^{-45\pi}) = \frac{3+\sqrt{5} + (\sqrt{3}+\sqrt{5}+\sqrt[4]{60})\sqrt[3]{2+\sqrt{3}}}{3\sqrt{10}+\sqrt{5}}\,\phi(e^{-\pi})$$

$$\frac{B(1-B)}{1+B} = u \quad \text{then} \quad \tfrac{1}{4}\left(u+\tfrac{1}{u}-2\right)$$

1 — 1
3 — 3
5 — 9
7 — $(\sqrt{2}+1)^2(1+2\sqrt{2})$
9 — 49
11 — 99
15 — $3(5+4\sqrt{2})^2$
23 — $9(\sqrt{2}+1)^4(3+4\sqrt{2})$
29 — 99^2
35 — $63(8\sqrt{2}+5\sqrt{5})^2$
71 — $9(\sqrt{2}+1)^{10}(2\sqrt{2}+1)^2(6\sqrt{2}+1)$.

$$F\ \frac{1-\sqrt{1-\frac{1}{4}\left(\frac{\sqrt{6+3\sqrt{3}}-\sqrt{2+3\sqrt{3}}}{2}\right)^{18}(\sqrt{3}+1)^6\left(2\sqrt{3\sqrt{3}+2}-3\sqrt{3\sqrt{3}-2}\right)^2}}{2} = e^{-\pi\sqrt{69}}.$$

$$F\ \frac{1-\sqrt{1-4\left(\frac{\sqrt{6+3\sqrt{3}}-\sqrt{2+3\sqrt{3}}}{2}\right)^{6}\left(\frac{\sqrt{3}+1}{2}\right)^6\left(2\sqrt{3\sqrt{3}+2}+3\sqrt{3\sqrt{3}-2}\right)^2}}{2} = e^{-\pi\sqrt{\frac{23}{3}}}.$$

$\sqrt{42}$ $(8-3\sqrt{7})^2(7-4\sqrt{3})^2(3-2\sqrt{2})^2(\sqrt{7}-\sqrt{6})^2$.

$\sqrt{78}$ $(2-\sqrt{3})^6(\sqrt{13}-2\sqrt{3})^4(3\sqrt{3}-\sqrt{26})^2(5-2\sqrt{6})^2$.

157 ~~180~~ 169

F $\dfrac{1-\sqrt{1-\left(\sqrt{\frac{9+\sqrt{73}}{8}}-\sqrt{\frac{1+\sqrt{73}}{8}}\right)^{24}}}{2} = e^{-\pi\sqrt{73}}$

~~F $\dfrac{1-\sqrt{1-\left(\sqrt{\frac{15+\sqrt{193}}{4}}-\sqrt{\frac{11+\sqrt{193}}{4}}\right)^{24}}}{2} = e^{-\pi\sqrt{193}}$~~

F $\dfrac{1-\sqrt{1-\dfrac{\left(\sqrt[3]{1+\sqrt{\frac{43}{27}}}+\sqrt[3]{1-\sqrt{\frac{43}{27}}}\right)^{24}}{210}}}{2} = e^{-\pi\sqrt{43}}$

$1+\dfrac{x^{\frac{1}{5}}}{1-x^{\frac{1}{5}}}+\dfrac{2x^{\frac{2}{5}}}{1-x^{\frac{2}{5}}}+\&c$

$-25\left(\dfrac{x^5}{1-x^5}+\dfrac{2x^{10}}{1-x^{10}}+\&c\right)$

$=\dfrac{f^5(-x)}{f(-x^5)}\sqrt{1+2x^{\frac{1}{5}}\dfrac{f(-x^5)}{f(-x^{\frac{1}{5}})}+5x^{\frac{2}{5}}\dfrac{f^2(-x^5)}{f^2(-x^{\frac{1}{5}})}}$

$\sqrt{102}$ $\left(\dfrac{\sqrt{51}-7}{\sqrt{2}}\right)^4(5-2\sqrt{6})^4(\sqrt{51}-5\sqrt{2})^2(2-\sqrt{3})^4.$

$(\sqrt{26}-5)^4(\sqrt{10}-3)^4.$

$\sqrt{130}$ $(5\sqrt{130}-57)^2(3-2\sqrt{2})^4(\sqrt{26}-5)^4$

$\sqrt{190}$ $\left(\dfrac{3\sqrt{19}-13}{\sqrt{2}}\right)^4(37\sqrt{19}-51\sqrt{10})^2(2\sqrt{5}-\sqrt{19})^4(\sqrt{19}-3\sqrt{2})^4$

δ

$$F\ \frac{1-\sqrt{1-\left(\sqrt{\frac{6+3\sqrt3}{4}}-\sqrt{\frac{2+3\sqrt3}{4}}\right)^{12}t}}{2}=e^{-\pi\sqrt{69}}$$

when $$t=\left(\sqrt{\frac{(5+3\sqrt3)\sqrt{4+6\sqrt3}+1}{2}}\pm\sqrt{\frac{(5+3\sqrt3)\sqrt{4+6\sqrt3}-1}{2}}\right)^4.$$

$$4\beta(1-\beta)=\left(\frac{\sqrt{13}-3}{2}\right)^6(\sqrt{13}-\sqrt{12})^4\left(\sqrt{\frac{11+6\sqrt3}{2}}\pm\sqrt{\frac{9+6\sqrt3}{2}}\right)^8$$ for $e^{-3\pi\sqrt{13}}$.

$$4\beta(1-\beta)=(\sqrt{37}-6)^6(2\sqrt{37}-7\sqrt3)^4\left(\sqrt{73+42\sqrt3}\pm\sqrt{72+42\sqrt3}\right)^8$$ for $e^{-3\pi\sqrt{37}}$.

$$4\beta(1-\beta)=\left(\frac{\sqrt[3]{2(\sqrt3-1)}-1}{\sqrt[3]{2(\sqrt3+1)}+1}\right)^8$$ for $e^{-9\pi}$.

$$4\beta(1-\beta)=\frac14\cdot\left\{\frac{1\pm\left(2\sqrt[6]{\frac{28}{27}}-\sqrt{\frac73}\right)}{2}\right\}^{24}$$ for $e^{-7\pi\sqrt3}$.

$$4\beta(1-\beta)=\frac{x^{24}}{2^{10}}$$ for $e^{-11\pi\sqrt3}$ when

$$x^3+x^2(1\pm\sqrt{11+2\sqrt{33}})+x(4+\sqrt{33}\pm\sqrt{11+2\sqrt{33}})-2=0$$

$\sqrt{217}$ $$\left(\sqrt{\frac{11+4\sqrt7}{2}}-\sqrt{\frac{9+4\sqrt7}{2}}\right)^{12}\left(\sqrt{\frac{16+5\sqrt7}{4}}\pm\sqrt{\frac{12+5\sqrt7}{4}}\right)^{12}$$

$\sqrt{205}$ $$(\sqrt5-2)^8\left(\frac{3\sqrt5-\sqrt{41}}{2}\right)^6\left(\sqrt{\frac{7+\sqrt{41}}{8}}\pm\sqrt{\frac{\sqrt{41}-1}{8}}\right)^{24}$$

$\sqrt{265}$ $$\left(\frac{\sqrt{53}\pm7}{2}\right)^6(\sqrt5\pm2)^6\left(\sqrt{\frac{89+5\sqrt{265}}{8}}-\sqrt{\frac{81+5\sqrt{265}}{8}}\right)^{12}$$

$\sqrt{57}$. $\frac{1}{4}(3\sqrt{19}-13)^4(2\pm\sqrt{3})^6$

$\sqrt{93}$. $\frac{1}{4}(39-7\sqrt{31})^4\left(\frac{\sqrt{31}\pm3\sqrt{3}}{2}\right)^6$.

$\sqrt{177}$ $\frac{1}{4}(3\sqrt{59}\pm23)^4(2-\sqrt{3})^{18}$.

$\sqrt{85}$ $(\sqrt{5}\pm2)^8\left(\frac{\sqrt{85}-9}{2}\right)^6$.

$\sqrt{133}$ $(8-3\sqrt{7})^6\left(\frac{5\sqrt{7}\pm3\sqrt{19}}{2}\right)^6$.

$\sqrt{55}$ $\frac{1}{64}(\sqrt{5}-2)^4\left(\sqrt{\frac{7+\sqrt{5}}{8}}\pm\sqrt{\frac{\sqrt{5}-1}{8}}\right)^{24}$

$\sqrt{65}$ $\left(\frac{\sqrt{13}\pm3}{2}\right)^6(\sqrt{5}\pm2)^2\left(\sqrt{\frac{9+\sqrt{65}}{8}}-\sqrt{\frac{1+\sqrt{65}}{8}}\right)^{12}$

$\sqrt{253}$ $(24-5\sqrt{23})^6\left(\frac{9\sqrt{23}\pm13\sqrt{11}}{2}\right)^6$.

$\sqrt{145}$ $(\sqrt{5}-2)^6\left(\frac{\sqrt{29}-5}{2}\right)^6\left(\sqrt{\frac{17+\sqrt{145}}{8}}\pm\sqrt{\frac{9+\sqrt{145}}{8}}\right)^{12}$

$\sqrt{117}$. $\left(\frac{\sqrt{13}-3}{2}\right)^6(\sqrt{13}-2\sqrt{3})^4\left(\frac{\sqrt{4+\sqrt{3}}\pm\sqrt[4]{3}}{2}\right)^{24}$.

$\sqrt{333}$ $(\sqrt{37}-6)^6(2\sqrt{37}-7\sqrt{3})^4\left(\frac{\sqrt{7+2\sqrt{3}}\pm\sqrt{3+2\sqrt{3}}}{2}\right)^{24}$.

$\sqrt{153}$ $\left(\sqrt{\frac{5+\sqrt{17}}{8}}-\sqrt{\frac{\sqrt{17}-3}{8}}\right)^{48}\left(\sqrt{\frac{37+9\sqrt{17}}{4}}\pm\sqrt{\frac{33+9\sqrt{17}}{4}}\right)$

$\sqrt{77}$ $(8\pm3\sqrt{7})^3\left(\frac{\sqrt{11}\pm\sqrt{7}}{2}\right)^3\left(\sqrt{\frac{6+\sqrt{11}}{4}}-\sqrt{\frac{2+\sqrt{11}}{4}}\right)^{12}$

$\sqrt{69}$ $\left(\frac{5\pm\sqrt{23}}{\sqrt{2}}\right)^2\left(\frac{3\sqrt{3}\pm\sqrt{23}}{2}\right)^3\left(\sqrt{\frac{6+3\sqrt{3}}{4}}-\sqrt{\frac{2+3\sqrt{3}}{4}}\right)$

$\sqrt{213}$ $\left(\frac{59\pm7\sqrt{71}}{\sqrt{2}}\right)^2\left(\frac{5\sqrt{3}\pm\sqrt{71}}{2}\right)^3\left(\sqrt{\frac{21+12\sqrt{3}}{2}}-\sqrt{\frac{19+12\sqrt{3}}{2}}\right)$

$$(1-e^{-\pi\sqrt{n}})(1-e^{-3\pi\sqrt{n}})(1-e^{-5\pi\sqrt{n}})\ \&c$$

$$= \sqrt[4]{2}\,.\,e^{-\frac{\pi}{24}\sqrt{n}}\,.\,g_n.$$

$$g_2 = 1;\quad g_6 = \sqrt[6]{1+\sqrt{2}};\quad g_{10} = \sqrt{\frac{1+\sqrt{5}}{2}};\quad g_{14} = \sqrt{\frac{3+\sqrt{2}}{4}} + \sqrt{\frac{\sqrt{2}-1}{4}}$$

$$g_{18} = \sqrt[3]{\sqrt{2}+\sqrt{3}};\quad g_{22} = \sqrt{1+\sqrt{2}};\quad g_{30} = \sqrt{\frac{1+\sqrt{5}}{2}}\ \sqrt[6]{3+\sqrt{10}}\,..$$

$$g_{58} = \sqrt{\frac{5+\sqrt{29}}{2}};\quad g_{70} = \frac{1+\sqrt{5}}{2}\sqrt{1+\sqrt{2}}\,.\quad g_{46} = \sqrt{\frac{5+\sqrt{2}}{4}} + \sqrt{\frac{1+\sqrt{2}}{4}}$$

$$g_{142} = \sqrt{\frac{11+5\sqrt{2}}{4}} + \sqrt{\frac{7+5\sqrt{2}}{4}}\,.\quad g_{82} = \sqrt{\frac{13+\sqrt{41}}{8}} + \sqrt{\frac{5+\sqrt{41}}{8}}$$

$$\sqrt{175}\,.\ \frac{\left\{\frac{\sqrt{5}-1}{2} + \sqrt[3]{\frac{5-\sqrt{5}}{4}}\left(\sqrt[3]{8-3\sqrt{5}+3\sqrt{21}} + \sqrt[3]{8-3\sqrt{5}-3\sqrt{21}}\right)\right\}^{24}}{64\,.\,3^{24}}$$

$$\sqrt{\frac{25}{7}}\ \frac{\left\{-\frac{\sqrt{5}+1}{2} + \sqrt[3]{\frac{5+\sqrt{5}}{4}}\left(\sqrt[3]{8+3\sqrt{5}+3\sqrt{21}} + \sqrt[3]{8+3\sqrt{5}-3\sqrt{21}}\right)\right\}}{64\,.\,3^{24}}$$

~~$$\sqrt{289}\,.\ \sqrt{\frac{17+\sqrt{17}+\sqrt{170}+42\sqrt{17}}{16}}$$~~

~~$$\sqrt{1+\sqrt{17}+\sqrt{170}+42}$$~~

$$g_{42} = \sqrt{\frac{7+\sqrt{3}}{2}}\ \sqrt[6]{2\sqrt{2}+\sqrt{7}}\,.\quad g_{78} = \sqrt{\frac{3+\sqrt{13}}{2}}\ \sqrt[6]{5+\sqrt{26}}$$

$$g_{102} = \sqrt{1+\sqrt{2}}\ \sqrt[3]{3\sqrt{2}+\sqrt{17}}\,.\quad g_{130} = \sqrt{\frac{3+\sqrt{13}}{2}}\ \sqrt{2+\sqrt{5}}$$

$$g_{190} = \sqrt{2+\sqrt{5}}\ \sqrt{3+\sqrt{10}}\,.\quad g_{34} = \sqrt{\frac{7+\sqrt{17}}{8}} + \sqrt{\frac{\sqrt{17}-1}{8}}$$

$\sqrt{[illegible]}$. $\left\{\sqrt{\frac{17+\sqrt{17}+(5+\sqrt{17})\sqrt[4]{17}}{16}} - \sqrt{\frac{1+\sqrt{17}+(5+\sqrt{17})\sqrt[4]{17}}{16}}\right\}^{48}$.

$\sqrt{121}$. $\left\{\frac{\sqrt[3]{11-3\sqrt{11}}\left(\sqrt[3]{3\sqrt{11}+3\sqrt{3}-4}+\sqrt[3]{3\sqrt{11}-3\sqrt{3}-4}\right)-2}{3\sqrt{2}}\right\}^{24}$.

$\sqrt{169}$ $\left\{\frac{\sqrt[3]{\frac{13-3\sqrt{13}}{2}}\left(\sqrt[3]{3\sqrt{3}-\frac{11-\sqrt{13}}{2}}-\sqrt[3]{3\sqrt{3}+\frac{11-\sqrt{13}}{2}}\right)+(\sqrt{13}-2)}{3}\right\}^{24}$

$\sqrt{105}$ $\left(\frac{5-\sqrt{21}}{2}\right)^{6}(2\pm\sqrt{3})^{6}(\sqrt{5}\pm 2)^{4}(6\pm\sqrt{35})^{2}$.

$\sqrt{165}$ $(4-\sqrt{15})^{6}(3\sqrt{5}\pm 2\sqrt{11})^{4}\left(\frac{\sqrt{15}\pm\sqrt{11}}{2}\right)^{6}(\sqrt{5}\pm 2)^{4}$.

$\sqrt{345}$. $\left(\frac{3\sqrt{3}-\sqrt{23}}{2}\right)^{12}\left(\frac{7\sqrt{23}\pm 15\sqrt{5}}{\sqrt{2}}\right)^{4}(\sqrt{5}\pm 2)^{8}(2\pm\sqrt{3})^{6}$

$\sqrt{385}$ $(10-3\sqrt{11})^{6}(6\pm\sqrt{35})^{6}\left(\frac{\sqrt{11}\pm\sqrt{7}}{2}\right)^{12}(\sqrt{5}\pm 2)^{8}$

$\sqrt{273}$ $\left(\frac{15\sqrt{7}-11\sqrt{13}}{\sqrt{2}}\right)^{4}\left(\frac{\sqrt{13}\pm 3}{2}\right)^{12}\left(\frac{\sqrt{7}\pm\sqrt{3}}{2}\right)^{12}(2\pm\sqrt{3})^{6}$.

$\sqrt{357}$ $\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{24}(8\pm 3\sqrt{7})^{6}\left(\frac{11\pm\sqrt{119}}{\sqrt{2}}\right)^{4}\left(\frac{\sqrt{21}\pm\sqrt{17}}{2}\right)^{6}$.

$$g_{98} = \left(\sqrt{\frac{\sqrt{2}\ 4 + \sqrt{14 + 4\sqrt{14}}}{8}} + \sqrt{\frac{\sqrt{2} - 4 + \sqrt{14 + 4\sqrt{14}}}{8}}\right)^2.$$

$$g_{90} = \sqrt{\frac{\sqrt{5}+1}{2}}\ \sqrt[6]{\sqrt{6}+\sqrt{5}}\left(\sqrt{\frac{3+\sqrt{6}}{4}} + \sqrt{\frac{\sqrt{6}-1}{4}}\right).$$

$$g_{198} = \sqrt{\sqrt{2}+1}\ \sqrt[6]{4\sqrt{2}+\sqrt{33}}\left(\sqrt{\frac{9+\sqrt{33}}{8}} + \sqrt{\frac{1+\sqrt{33}}{8}}\right)$$

If $g_n^4 - \dfrac{1}{g_n^4} = p$, then $g_{9n} = g_n \sqrt[6]{p + \sqrt{p^2+1}}\ \times$

$$\sqrt[3]{\left\{\sqrt{\frac{p^2+4+\sqrt{(p^2+1)(p^2+4)}}{2}} + \sqrt{\frac{p^2+2+\sqrt{(p^2+1)(p^2+4)}}{2}}\right\}}.$$

$$g_{522} = \sqrt{\frac{5+\sqrt{29}}{2}}\ \sqrt[6]{5\sqrt{29}+11\sqrt{6}}\left(\sqrt{\frac{9+3\sqrt{6}}{4}} + \sqrt{\frac{5+3\sqrt{6}}{4}}\right)$$

$$g_{630} = \frac{1+\sqrt{5}}{2}\sqrt{1+\sqrt{2}}\ \sqrt[6]{\sqrt{15}+\sqrt{14}}\ \sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}}\left(\sqrt{\frac{2+\sqrt{7}+\sqrt{15}}{4}} + \sqrt{\frac{\sqrt{7}+\sqrt{15}-2}{4}}\right)$$
$$\times\left(\sqrt{\frac{4+\sqrt{7}+\sqrt{15}}{8}} + \sqrt{\frac{\sqrt{7}+\sqrt{15}-4}{8}}\right).$$

$$g_{1170} = \sqrt{2+\sqrt{5}}\ \sqrt{\frac{3+\sqrt{13}}{2}}\ \sqrt{\sqrt{2}+\sqrt{3}}\ \sqrt[6]{2\sqrt{10}+\sqrt{39}}.$$

$$g_{50} = \frac{1 + \sqrt[3]{\frac{5+\sqrt{5}}{4}}\left(\sqrt[3]{1+7\sqrt{5}+6\sqrt{6}} + \sqrt[3]{1+7\sqrt{5}-6\sqrt{6}}\right)}{3}.$$

$$g_{126} = \sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}}\ \sqrt[6]{\sqrt{7}+\sqrt{6}}\left(\sqrt{\frac{3+\sqrt{2}}{4}} + \sqrt{\frac{\sqrt{2}-1}{4}}\right)^2.$$

$$g_{26} = \frac{1}{3}\left\{\sqrt{2+\sqrt{13}} + \sqrt[3]{(2+\sqrt{13})\sqrt{2+\sqrt{13}} + 3\sqrt{3(3+\sqrt{13})}} + \sqrt[3]{(2+\sqrt{13})\sqrt{2+\sqrt{13}} - 3\sqrt{3(3+\sqrt{13})}}\right\}$$

$$g_{66} = \sqrt[4]{\sqrt{2}+\sqrt{3}}\ \sqrt[12]{3\sqrt{11}+7\sqrt{2}}\ \sqrt{\sqrt{\frac{7+\sqrt{33}}{8}} \pm \sqrt{\frac{\sqrt{33}-1}{8}}}$$ ~~172~~ 160

$$g_{138} = \sqrt[4]{\frac{3\sqrt{3}+\sqrt{23}}{2}}\ \sqrt[12]{23\sqrt{23}+78\sqrt{2}}\ .\ \sqrt{\sqrt{\frac{5+2\sqrt{6}}{4}} \pm \sqrt{\frac{1+2\sqrt{6}}{4}}}\ .$$

~~$$g_{154} = \sqrt[4]{2\sqrt{2}\pm\sqrt{7}}\ \sqrt[4]{\frac{\sqrt{11}\pm\sqrt{7}}{2}}\ .\ \sqrt{\sqrt{\frac{9+2\sqrt{22}}{4}} \pm \sqrt{\frac{5+2\sqrt{22}}{4}}}$$~~

$$g_{114} = \sqrt[4]{\sqrt{3}\pm\sqrt{2}}\ \sqrt[12]{\sqrt{19}\pm 3\sqrt{2}}\ \sqrt{\frac{3\sqrt{3}+\sqrt{19}}{4} + \sqrt{\frac{15+3\sqrt{57}}{8}}}\ .$$

$$g_{238} = \left(\sqrt{\frac{5+3\sqrt{2}}{4}} + \sqrt{\frac{1+3\sqrt{2}}{4}}\right)\left(\sqrt{\frac{5+2\sqrt{2}}{4}} \pm \sqrt{\frac{1+2\sqrt{2}}{4}}\right).$$

$$g_{62} = \left(\sqrt{\frac{4+\sqrt{1+\sqrt{2}}+\sqrt{9+5\sqrt{2}}}{8}} + \sqrt{\frac{\sqrt{1+\sqrt{2}}+\sqrt{9+5\sqrt{2}}-4}{8}}\right)^2$$

$$g_{94} = \left(\sqrt{\frac{4+\sqrt{7+\sqrt{2}}+\sqrt{7+5\sqrt{2}}}{8}} + \sqrt{\frac{\sqrt{7+\sqrt{2}}+\sqrt{7+5\sqrt{2}}-4}{8}}\right)^2$$

$$g_{154} = \sqrt[4]{2\sqrt{2}\pm\sqrt{7}}\ \sqrt[4]{\frac{\sqrt{11}\pm\sqrt{7}}{2}}\ \sqrt{\sqrt{\frac{13+2\sqrt{22}}{4}} + \sqrt{\frac{9+2\sqrt{22}}{4}}}$$

$$g_{310} = \frac{\sqrt{5}\pm 1}{2}\ \sqrt{\sqrt{2}\pm 1}\ \left(\sqrt{\frac{7+2\sqrt{10}}{4}} + \sqrt{\frac{3+2\sqrt{10}}{4}}\right)$$

$$g_{158} = \left(\sqrt{\frac{4+\sqrt{9+\sqrt{2}}+\sqrt{17+13\sqrt{2}}}{8}} + \sqrt{\frac{\sqrt{9+\sqrt{2}}+\sqrt{17+13\sqrt{2}}-4}{8}}\right)^2.$$

$$\sqrt{465}\quad \left(\sqrt{\frac{13+2\sqrt{31}}{2}} - \sqrt{\frac{11+2\sqrt{31}}{2}}\right)^{12}\left(\frac{\sqrt{31}-3\sqrt{3}}{2}\right)^{6}(2-\sqrt{3})^{6}$$

$$\times\left(\sqrt{\frac{6+\sqrt{31}}{4}} - \sqrt{\frac{2+\sqrt{31}}{4}}\right)^{12}(5\sqrt{5}-2\sqrt{31})^{2}(\sqrt{5}-2)^{2}$$

$$\sqrt{777}\quad (\sqrt{37}-6)^{6}(107\sqrt{37}-246\sqrt{7})^{2}\left(\sqrt{\frac{17+6\sqrt{7}}{2}} - \sqrt{\frac{15+6\sqrt{7}}{2}}\right)^{12}$$

$$\times\left(\sqrt{\frac{10+3\sqrt{7}}{4}} - \sqrt{\frac{6+3\sqrt{7}}{4}}\right)^{12}(2-\sqrt{3})^{6}\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{6}$$

$$\sqrt{1353}\quad \left(\sqrt{\frac{569+99\sqrt{33}}{8}} - \sqrt{\frac{561+99\sqrt{33}}{8}}\right)^{12}\left(\frac{321\sqrt{451}-6817}{\sqrt{2}}\right)^{2}$$

$$\times\left(\frac{\sqrt{123}-11}{\sqrt{2}}\right)^{6}(10-3\sqrt{11})^{3}(2-\sqrt{3})^{9}\left(\sqrt{\frac{25+3\sqrt{33}}{8}} - \sqrt{\frac{17+3\sqrt{33}}{8}}\right)^{12}$$

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$\sqrt{1645}$ $$\left(\sqrt{\frac{751+41\sqrt{329}}{8}}-\sqrt{\frac{743+41\sqrt{329}}{8}}\right)^{12}\left(\frac{9\sqrt{329}-73\sqrt{5}}{2}\right)^{3}$$
$$\times(\sqrt{5}-2)^{12}\left(\sqrt{\frac{127+7\sqrt{329}}{8}}-\sqrt{\frac{119+7\sqrt{329}}{8}}\right)^{12}\left(\frac{7-\sqrt{47}}{\sqrt{2}}\right)^{6}(8-3\sqrt{7})^{3}$$

$\sqrt{897}$ $\quad Q+\frac{1}{Q}=58+9\sqrt{39} \qquad s+\frac{1}{s}=6+\sqrt{39}$

$\sqrt{1677}$ $\quad s+\frac{1}{s}=15+2\sqrt{43}$

$\sqrt{141}$ $$(4\sqrt{3}\pm\sqrt{47})^{3}\left(\frac{7\pm\sqrt{47}}{\sqrt{2}}\right)^{2}\left(\sqrt{\frac{18+9\sqrt{3}}{4}}-\sqrt{\frac{14+9\sqrt{3}}{4}}\right)^{12}$$

$\sqrt{445}$ $$(\sqrt{5}-2)^{12}\left(\frac{\sqrt{445}-21}{2}\right)^{6}\left(\sqrt{\frac{13+\sqrt{89}}{8}}\pm\sqrt{\frac{5+\sqrt{89}}{8}}\right)^{24}$$

$\sqrt{553}$ $$\left(\sqrt{\frac{143+16\sqrt{79}}{2}}-\sqrt{\frac{141+16\sqrt{79}}{2}}\right)^{12}\left(\sqrt{\frac{100+11\sqrt{79}}{4}}\pm\sqrt{\frac{96+11\sqrt{79}}{4}}\right)^{12}$$

$$g_{210}=\sqrt{\sqrt{3}+\sqrt{2}}\;\sqrt[6]{3\sqrt{14}+5\sqrt{5}}\;\sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}}\;\sqrt{\frac{\sqrt{5}+1}{2}}$$

$$g_{330}=\sqrt{\sqrt{6}+\sqrt{5}}\;\sqrt{\frac{\sqrt{15}+\sqrt{11}}{2}}\cdot\frac{\sqrt{5}+1}{2}\cdot\sqrt[6]{\sqrt{11}+\sqrt{10}}.$$

~~If u & v are~~ If $g_n^{6}=uv$; $u^2+\frac{1}{u^2}=2U$, $v^2+\frac{1}{v^2}=2V$,

$\sqrt{U^2+V^2-1}=W$ and $2S=U+V+W+1$, then

$$F\;(\sqrt{S}-\sqrt{S-1})^{2}(\sqrt{S-U}-\sqrt{S-U-1})^{2}(\sqrt{S-V}-\sqrt{S-V-1})^{2}(\sqrt{S-W}-\sqrt{S-W-1})^{2}=e^{-\pi\sqrt{n}}.$$

$\sqrt{210}$:—
$$(4-\sqrt{15})^{4}(\sqrt{10}-3)^{4}(\sqrt{7}-\sqrt{6})^{4}(8-3\sqrt{7})^{2}(6-\sqrt{35})^{2}$$
$$\times(\sqrt{15}-\sqrt{14})^{2}(3-2\sqrt{2})^{2}(2-\sqrt{3})^{2}.$$

If $2u = 11 + \frac{f^6(-x)}{x f^6(-x^5)}$ and $2v = 1 + \frac{f(-x^{\frac{1}{5}})}{x^{\frac{1}{5}} f(-x^5)}$

then $\sqrt[5]{\sqrt{u^2+1} - u} = \sqrt{v^2+1} - v = \frac{\sqrt[5]{x}}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+}\,\&c$

$$3 + \frac{f^3(-x^{\frac{1}{3}})}{x^{\frac{1}{3}} f^3(-x^3)} = \sqrt[3]{27 + \frac{f^{12}(-x)}{x f^{12}(-x^3)}}$$

$$f^3(-x^{\frac{1}{3}}) + 3x^{\frac{1}{3}} f^3(-x^3) = f(-x)\left\{1 + 6\left(\frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} - \&c\right)\right\}$$

$$1 + \frac{\psi(-x^{\frac{1}{3}})}{x^{\frac{1}{3}}\psi(-x^3)} = \sqrt[3]{1 + \frac{\psi^4(-x)}{x\psi^4(-x^3)}}$$

$$\frac{\phi(x^{\frac{1}{3}})}{\phi(x^3)} - 1 = \sqrt[3]{\frac{\phi^4(x)}{\phi^4(x^3)} - 1}$$

$$1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.1}{2.4}\right)^2 x^2 + \left(\frac{1.1.3}{2.4.6}\right)^2 x^3 + \&c$$
$$= \frac{7}{3}(1+x) + \frac{2}{3z}\left\{1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)\right\}$$

$$1 - \frac{1^2}{2^2}x - \frac{1^2.3}{2^2.4^2}x^2 - \frac{1^2.3^2.5}{2^2.4^2.6^2}x^3 - \&c$$
$$= \frac{7}{3}(2-x) + \frac{1}{3z}\left\{1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)\right\}$$

$$\frac{\cos\theta + 2\cos\frac{\theta}{2}\cosh\frac{\theta}{2}\sqrt{3}}{\cosh\frac{\pi\sqrt{3}}{2}} - \frac{\cos 3\theta + 2\cos\frac{3\theta}{2}\cosh\frac{3\theta\sqrt{3}}{2}}{3\cosh\frac{3\pi\sqrt{3}}{2}} + \&c = \frac{\pi}{8}.$$

$$\frac{1^5}{1^6 - x^6}\,\frac{1}{\cosh\frac{\pi\sqrt{3}}{2}} - \frac{3^5}{3^6 - x^6}\,\frac{1}{\cosh\frac{3\pi\sqrt{3}}{2}} + \&c$$
$$= \frac{\pi}{12}\cdot\frac{1}{\cos\frac{\pi x}{2}\left\{\cos\frac{\pi x}{2} + \cosh\frac{\pi x\sqrt{3}}{2}\right\}}$$

If $u = \frac{f^4(-x)}{x f^4(-x^7)}$ and $v = \frac{f(-x^{\frac{1}{7}})}{x^{\frac{2}{7}} f(-x^7)}$, then

$$2u = 7(v^3 + 5v^2 + 7v) + (v^2 + 7v + 7)\sqrt{4v^3 + 21v^2 + 28v}$$

$$1 + 12\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \&c\right) - 12\left(\frac{3x^3}{1-x^3} + \frac{6x^6}{1-x^6} + \&c\right)$$

$$= \frac{\{\psi^4(x) + 3x\psi^4(x^3)\}^2}{\psi^2(x)\psi^2(x^3)} = \frac{\{f^{12}(-x) + 27x f^{12}(-x^3)\}^{\frac{2}{3}}}{f^2(-x) f^2(-x^3)}$$

$$= \left\{\frac{\phi^4(\sqrt{x}) + 3\phi^4(\sqrt{x^3})}{4\phi(\sqrt{x})\phi(\sqrt{x^3})}\right\}^2$$

$$1 + 12\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \&c\right) - 12\left(\frac{3x^6}{1-x^6} + \frac{6x^{12}}{1-x^{12}} + \&c\right)$$

$$= \left\{\frac{\phi^4(x) + 3\phi^4(x^3)}{4\phi(x)\phi(x^3)}\right\}^2 = \phi^2(x)\phi^2(x^3)\left\{1 - \frac{4x}{\chi^6(x)\chi^6(x^3)}\right\}$$

$$1 + 6\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \&c\right) - 6\left(\frac{5x^5}{1-x^5} + \frac{10x^{10}}{1-x^{10}} + \&c\right)$$

$$= \sqrt{f^{12}(-x) + 22x f^6(-x) f^6(-x^5) + 125x^2 f^{12}(-x^5)} \Big/ f(-x)f(-x^5).$$

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$$= \{\psi^4(x) + 2x\psi^2(x)\psi^2(x^5) + 5x^2\psi^4(x^5)\}\sqrt{\psi^4(x) - 2x\psi^2(x)\psi^2(x^5) + 5x^2\psi^4(x^5)}$$

$$\div\ \psi(x)\psi(x^5).$$

$$1 + 6\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \&c\right) - 6\left(\frac{5x^{10}}{1-x^{10}} + \frac{10x^{20}}{1-x^{20}} + \&c\right)$$

$$= \phi^2(x)\phi^2(x^5)\left\{1 - \frac{2x}{\chi^4(x)\chi^4(x^5)}\right\}\sqrt{1 - \frac{4x}{\chi^4(x)\chi^4(x^5)}}$$

$$1 + 4\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right) - 4\left(\frac{7x^7}{1-x^7} + \frac{14x^{14}}{1-x^{14}} + \&c\right)$$

$$= \left\{\frac{f^8(-x) + 13x f^4(-x) f^4(-x^7) + 49x^2 f^8(-x^7)}{f(-x)f(-x^7)}\right\}^{\frac{2}{3}}$$

$$1+4\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\right)-4\left(\frac{7x^{14}}{1-x^{14}}+\frac{14x^{28}}{1-x^{28}}+\&c\right)$$

$$=\phi^2(x)\phi^2(x^7)\left\{1-\frac{2x}{\chi^3(x)\chi^3(x^7)}\right\}^2.$$

$$1+3\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\&c\right)-3\left(\frac{9x^9}{1-x^9}+\frac{18x^{18}}{1-x^{18}}+\&c\right)$$

$$=\frac{f^6(-x^3)}{f^2(-x)f^2(-x^9)}\left\{f^6(-x)+9xf^3(-x)f^3(-x^9)+27x^2f^6(-x^9)\right\}^{\frac{1}{3}}.$$

$$\times\left\{\frac{\phi^4(x^3)+3\phi^2(x)\phi^2(x^9)}{4}\right\}^2\cdot\frac{\phi^2(x^3)}{\phi^3(x)\phi^3(x^9)}=1+3\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)-3\left(\frac{9x^{18}}{1-x^{18}}+\&c\right)$$

$$*\ \frac{f(x^2,-x^7)}{f(x,-x^8)}\ (u)=\frac{f(x^4,-x^5)}{f(x^2,-x^7)}\ (v)+x\frac{f(x,-x^8)}{f(-x^4,-x^5)}\ (w)$$

$$f(x^{\frac{1}{3}})=f(x^4,-x^5)-x^{\frac{1}{3}}f(-x^2,-x^7)-x^{\frac{2}{3}}f(-x,-x^8).$$

$$*\ -\frac{1}{u}+\frac{1}{v}+\frac{1}{w}=\frac{f^4(-x^3)}{f(-x)f^3(-x^9)}$$

$$5+12\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\frac{3x^6}{1-x^6}+\&c\right)-12\left(\frac{11x^{22}}{1-x^{22}}+\frac{22x^{44}}{1-x^{44}}+\&c\right)$$

$$=5\phi^2(x)\phi^2(x^{11})-20xf^2(x)f^2(x^{11})+32x^2f^2(-x^2)f^2(-x^{22})$$

$$-20x^3\psi^2(-x)\psi^2(-x^{11}).$$

$$=\phi^2(x)\phi^2(x^{11})\left[5.\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}-\frac{1}{2}\left\{1-\sqrt[4]{\alpha\beta}-\sqrt[4]{(1-\alpha)(1-\beta)}\right\}^2\right]$$

$$3+4\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)-4\left(\frac{19x^{38}}{1-x^{38}}+\frac{38x^{76}}{1-x^{76}}+\&c\right)$$

$$=\phi^2(x)\phi^2(x^{19})\left[3.\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}-\frac{1}{2}\left\{1-\sqrt[4]{\alpha\beta}-\sqrt[4]{(1-\alpha)(1-\beta)}\right\}^2\right]$$

$$11+12\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)-12\left(\frac{23x^{46}}{1-x^{46}}+\frac{46x^{92}}{1-x^{92}}+\&c\right)$$

$$=\phi^2(x)\phi^2(x^{23})\left\{11.\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}\right.$$

$$-16\sqrt[3]{2}.\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}.\frac{1+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}}{2}$$

$$\left.-16\sqrt[3]{4}.\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}\right\}$$

$$7 + 12\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \&c\right) - 12\left(\frac{15x^{30}}{1-x^{30}} + \frac{30x^{60}}{1-x^{60}} + \&c\right)$$

$$= \phi^2(x)\,\phi^2(x^{15})\left\{7\cdot\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}\right.$$

$$\left. - 4\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\left\{1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}\right\}\right\}$$

$$\frac{x}{x+}\;\frac{x+1}{x+1+}\;\frac{\ }{x+2} \quad - \frac{8\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}}{\left(1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}\right)^2}\Big\}$$

$$5 + 4\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \&c\right) - 4\left(\frac{31x^{62}}{1-x^{62}} + \frac{62x^{124}}{1-x^{124}} + \&c\right)$$

$$= \phi^2(x)\,\phi^2(x^{31})\left\{8\cdot\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2} - 6\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\left(1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}\right)\right.$$

$$\left. - 4\sqrt[16]{\alpha\beta(1-\alpha)(1-\beta)}\sqrt{1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}}\left(1+\sqrt[4]{\alpha\beta}+\sqrt[4]{1-\alpha.1-\beta}\right)\right.$$

$$1 + 6\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \&c\right) - 6\left(\frac{5x^{10}}{1-x^{10}} + \frac{10x^{20}}{1-x^{20}} + \&c\right)$$

$$= \phi^2(x)\,\phi^2(x^5)\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{1-\alpha.1-\beta}}{2}}\left\{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2} + \frac{1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}}{4}\right.$$

~~$= \phi^2(x)\,\phi^2(x^5)\sqrt{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}$~~

$$= \phi^2(x)\,\phi^2(x^5)\sqrt{\frac{1+\alpha\beta+(1-\alpha)(1-\beta)}{2}} - \frac{3}{\sqrt[3]{4}}\sqrt[3]{\alpha\beta(1-\alpha)(1-\beta)}$$

$$v = \frac{\sqrt[3]{x}}{1+}\;\frac{x+x^2}{1+}\;\frac{x^2+x^4}{1+}\;\frac{x^3+x^6}{1+\&c} = \sqrt[3]{x}\,\frac{\chi(-x)}{\chi^3(-x^3)}$$

$$\frac{1}{v} + 1 = \frac{\psi(x^{\frac{1}{3}})}{x^{\frac{1}{3}}\psi(x^3)}; \quad \frac{1}{v^3} + 1 = \frac{\psi^4(x)}{x\,\psi^4(x^3)}.$$

$$= \sqrt{x}\,\frac{f(-x,-x^7)}{f(-x^3,-x^5)}$$

$$v = \frac{\sqrt{x}}{1+x+}\;\frac{x^2}{1+x^3+}\;\frac{x^4}{1+x^5+}\;\frac{x^6}{1+x^7+\&c}$$

$$\frac{1}{v} - v = \frac{\phi(x^2)}{\sqrt{x}\,\psi(x^4)}; \quad \frac{1}{v} + v = \frac{\phi(x)}{\sqrt{x}\,\psi(x^4)}$$

$$(8 \pm 3\sqrt{7})^3 \left(\frac{23\sqrt{43} \pm 57\sqrt{7}}{2}\right)^3 \times$$

$$\left(\sqrt{\frac{46+7\sqrt{43}}{4}} - \sqrt{\frac{42+7\sqrt{43}}{4}}\right)^{12}$$

$$f(-x^7, -x^8) + x f(-x^2, -x^{13}) = \frac{f(-x^2, -x^3)}{f(-x, -x^4)} f(-x^5).$$

$$[illegible](-x^4, -x^{11}) - x f(-x, -x^{14}) = \frac{f(-x, -x^4)}{f(-x^2, -x^3)} f(-x^5).$$

$$[illegible](-x, -x^{14}) f(-x^4, -x^{11}) f(-x^6, -x^9) f(-x^5)$$

$$= f(-x, -x^4) f^3(-x^{15}).$$

$$f(-x^7, -x^8) f(-x^2, -x^{13}) f(-x^3, -x^{12}) f(-x^5)$$

$$= f(-x^2, -x^3) f^3(-x^{15}).$$

$$f(-x^7, -x^8) - x f(-x^2, -x^{13}) = f(-x^{\frac{2}{3}}, -x) + x^{\frac{2}{3}} f(-x^3, -x^{12})$$

$$f(-x^4, -x^{11}) + x f(-x, -x^{14}) = \frac{f(-x^6, -x^9) - f(-x^{\frac{1}{3}}, -x^{\frac{4}{3}})}{x^{\frac{1}{3}}}.$$

$$\frac{\sqrt[8]{x}}{1+}\ \frac{x}{1+x+}\ \frac{x^2}{1+x^2+}\ \frac{x^3}{1+x^3+}\ \frac{x^4}{1+x^4+}\ \&c = \sqrt[8]{x} \cdot \frac{\psi(x)}{\phi(x)}.$$

$$\frac{f(-x^{\frac{1}{7}})}{f(-x^7)} = \frac{f(-x^2, -x^5)}{f(-x, -x^6)} - x^{\frac{1}{7}} \frac{f(-x^3, -x^4)}{f(-x^2, -x^5)} - x^{\frac{2}{7}} + x^{\frac{5}{7}} \frac{f(-x, -x^6)}{f(-x^3, -x^4)},$$

$$\frac{f(-x^{\frac{1}{11}})}{f(-x^{11})} = \frac{f(-x^4, -x^7)}{f(-x^2, -x^9)} - x^{\frac{1}{11}} \frac{f(-x^2, -x^9)}{f(-x, -x^{10})} - x^{\frac{2}{11}} \frac{f(-x^5, -x^6)}{f(-x^3, -x^8)}$$

$$+ x^{\frac{5}{11}} + x^{\frac{7}{11}} \frac{f(-x^3, -x^8)}{f(-x^4, -x^7)} - x^{\frac{15}{11}} \frac{f(-x, -x^{10})}{f(-x^5, -x^6)};$$

$$f(-x^{\frac{1}{7}}) + x^{\frac{2}{7}} f(-x^7) = \sqrt[7]{u} - \sqrt[7]{v} + \sqrt[7]{w}$$

$$\sqrt[7]{uvw} = x^{\frac{6}{7}} f^3(-x^7).$$

$$u - v + w = \frac{f^8(-x) + 14x f^4(-x) f^4(-x^7) + 57x^2 f^8(-x^7)}{f(-x^7)}$$

$$uv - uw + vw = f^2(-x^7)\left\{ f^{12}(-x) + 19x f^8(-x) f^4(-x^7) \right.$$

$$\left. + 126x^2 f^4(-x) f^8(-x^7) + 289x^3 f \text{[illegible]} \right.$$

$$\frac{f(-x^{\frac{1}{13}})}{x^{\frac{7}{13}} f(-x^{13})} = \frac{f(-x^4, -x^9)}{x^{\frac{7}{13}} f(-x^2, -x^{11})} - \frac{f(-x^6, -x^7)}{x^{\frac{6}{13}} f(-x^3, -x^{10})} - \frac{f(-x^2, -x^{11} \ldots}{x^{\frac{5}{13}} f(-x, \ldots}$$

$$+ \frac{f(-x^5, -x^8)}{x^{\frac{2}{13}} f(-x^4, -x^9)} + 1 - x^{\frac{5}{13}} \frac{f(-x^3, -x^{10})}{f(-x^5, -x^8)}$$

$$+ x^{\frac{15}{13}} \frac{f(-x, -x^{12})}{f(-x^6, -x^7)}.$$

$$= u_1 - u_2 - u_3 + u_4 + 1 - u_5 + u_6$$

$$u_1 u_2 - u_3 u_5 - u_4 u_6 = 1 + \frac{f^2(-x)}{x f^2(-x^{13})}.$$

$$u_1 u_2 \cdot u_3 u_5 - u_3 u_5 \cdot u_4 u_6 + u_4 u_6 \cdot u_1 u_2 = 4 + \frac{f^2(-x)}{x f^2(-x^{13})}$$

$$u_1 u_2 u_3 u_4 u_5 u_6 = 1.$$

$$u_2 u_3 u_4 - u_1 u_5 u_6 = 3 + \frac{f^2(-x)}{x f^2(-x^{13})}.$$

$$\frac{f(-x^{\frac{1}{17}})}{x^{\frac{12}{17}} f(-x^{17})} = \frac{f(-x^6,-x^{11})}{x^{\frac{12}{17}} f(-x^3,-x^{14})} - \frac{1}{x^{\frac{11}{17}}}\,\frac{f(-x^4,-x^{13})}{f(-x^2,-x^{15})}$$

$$-\frac{1}{x^{\frac{10}{17}}}\cdot\frac{f(-x^8,-x^9)}{f(-x^4,-x^{13})} + \frac{1}{x^{\frac{7}{17}}}\,\frac{f(-x^2,-x^{15})}{f(-x,-x^{16})}$$

$$+\frac{1}{x^{\frac{5}{17}}}\,\frac{f(-x^7,-x^{10})}{f(-x^5,-x^{12})} - 1 - x^{\frac{3}{17}}\,\frac{f(-x^5,-x^{12})}{f(-x^6,-x^{11})}$$

$$+x^{\frac{14}{17}}\,\frac{f(-x^3,-x^{14})}{f(-x^7,-x^{10})} - x^{\frac{28}{17}}\,\frac{f(-x,-x^{16})}{f(-x^8,-x^9)}$$

$$= u_1 - u_2 - u_3 + u_4 + u_5 - 1 - u_6 + u_7 - u_8$$

$$u_1 u_5 + u_2 u_8 - u_3 u_4 - u_6 u_7 = -1$$

$$u_1 u_5 \cdot u_6 u_7 = 1$$

$$u_2 u_8 \cdot u_3 u_4 = 1$$

1, 3, 9, 27 $\quad \dfrac{\phi(x^3)\,\phi(x^9)}{\phi(x)\,\phi(x^{27})} = \dfrac{\sqrt[4]{16\beta\gamma(1-\beta)(1-\gamma)} + \sqrt[8]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[4]{16\beta\gamma(1-\beta)(1-\gamma)} + \sqrt[8]{16\beta\gamma(1-\beta)(1-\gamma)}}$

1, 5, 7, 35, $\quad \dfrac{\phi(x^5)\,\phi(x^7)}{\phi(x)\,\phi(x^{35})} = \dfrac{\sqrt[4]{16\beta\gamma(1-\beta)(1-\gamma)} - \sqrt[8]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[4]{16\beta\gamma(1-\beta)(1-\gamma)} + \sqrt[8]{16\beta\gamma(1-\beta)(1-\gamma)}}$

$$= \frac{\sqrt[8]{16\alpha\delta(1-\alpha)(1-\delta)} + \sqrt[4]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[8]{16\beta\gamma(1-\beta)(1-\gamma)} - \sqrt[4]{16\alpha\delta(1-\alpha)(1-\delta)}}$$

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~~$\left\{1 - \sqrt[3]{\alpha\beta} - \sqrt[3]{(1-\alpha)(1-\beta)}\right\}^3.$~~

$$1 - \sqrt[3]{\alpha\beta} - \sqrt[3]{(1-\alpha)(1-\beta)} = P \qquad \text{~~$P + P_1 =$~~}$$

$$P_1 = \sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)}$$

$$9\sqrt[3]{\alpha\beta(1-\alpha)(1-\beta)} = R$$

~~8. $P^3 + 36PR - \frac{162R^2}{P} = 81R.$~~

~~8. $P^3 + 4PR - \frac{2R^2}{P} = 9R.$~~

$$P^3 - (5P + 9P_1)R - \frac{2R^2}{P} = 0$$

$$\frac{1}{2n} + \frac{n^2}{1(1^2+n^2)} + \frac{-n^2}{2(2^2+n^2)} + \frac{n^2}{3(3^2+n^2)} + \&c$$

$$+ \frac{4n^2}{1^4-n^4}\cdot\frac{1}{e^{2\pi}-1} + \frac{8n^2}{2^4-n^4}\cdot\frac{1}{e^{4\pi}-1} + \frac{12n^2}{3^4-n^4}\cdot\frac{1}{e^{6\pi}-1} + \&c$$

$$= \frac{1}{2\pi n^2} - \frac{\pi\cot\pi n}{e^{2\pi n}-1} + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

$$\frac{n^2}{1(1^2+n^2)} + \frac{n^2}{3(3^2+n^2)} + \frac{n^2}{5(5^2+n^2)} + \&c$$

$$+ \left(\frac{4n^2}{1^4-n^4}\cdot\frac{1}{e^{\pi}+1} + \frac{12n^2}{3^4-n^4}\cdot\frac{1}{e^{3\pi}+1} + \&c\right)$$

$$= -\frac{\pi}{2}\cdot\frac{\tan\frac{\pi n}{2}}{e^{\pi n}+1} + 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{n-1}$$

$$\frac{1}{2n} + 2n\left\{\frac{1}{(1^2-n^2)}\cdot\frac{1}{e^{\pi}-1} - \frac{1}{3^2-n^2}\cdot\frac{1}{e^{3\pi}-1} + \&c\right\}$$

$$+ 2n\left\{\frac{1}{(2^2+n^2)}\cdot\frac{1}{e^{\pi}+e^{-\pi}} + \frac{1}{4^2+n^2}\cdot\frac{1}{e^{2\pi}+e^{-2\pi}} + \&c\right\}$$

$$= \frac{\pi}{2}\cdot\frac{\sec\frac{\pi n}{2}}{e^{\pi n}-1} + \frac{1}{n+1} - \frac{1}{n+3} + \frac{1}{n+5} - \frac{1}{n+7} + \&c.$$

(1) $P = \dfrac{f(-x)}{x^{1/12} f(-x^3)}$ & $Q = \dfrac{f(-x^5)}{x^{5/12} f(-x^{15})}$. $(PQ)^2 + \left(\dfrac{3}{PQ}\right)^2 + 5 = \left(\dfrac{Q}{P}\right)^3 - \left(\dfrac{P}{Q}\right)^3$. ~~177~~ 165

(2) If $P = \dfrac{f(-x)}{x^{1/12} f(-x^3)}$ & $Q = \dfrac{f(-x^7)}{x^{7/12} f(-x^{21})}$, then

$$(PQ)^3 + \left(\frac{3}{PQ}\right)^3 = \left(\frac{Q}{P}\right)^4 - 7\cdot\left(\frac{Q}{P}\right)^2 + 7\left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^4.$$

(3) If $P = \dfrac{f(-x)}{x^{1/6} f(-x^5)}$ & $Q = \dfrac{f(-x^3)}{x^{1/2} f(-x^{15})}$

$$(PQ)^3 + \left(\frac{5}{PQ}\right)^3 = \left(\frac{Q}{P}\right)^6 - 9\cdot\left(\frac{Q}{P}\right)^3 - 9\cdot\left(\frac{P}{Q}\right)^3 - \left(\frac{P}{Q}\right)^6.$$

$$\frac{1}{2x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \&c$$

$$= \frac{1}{x} + \frac{1}{2x^2} \cdot \frac{1}{3x +} \frac{3}{5x +} \frac{18}{7x +} \frac{60}{9x +} \&c$$

$3 = 2^2(2^2-1)/4$

$18 = 3^2(3^2-1)/4$

$60 = 4^2(4^2-1)/4$ &c.

$$\frac{1}{2x^3} + \frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \&c$$

$$= \frac{1}{2x^2} + \frac{1}{4x^3} \cdot \frac{1}{x +} \frac{1}{3x +} \frac{2}{x +} \frac{6}{5x +} \frac{9}{x +} \frac{18}{7x +} \&c$$

$$\frac{x}{1+n+}\,\frac{1^2x^2}{3+n+}\,\frac{2^2x^2}{5+n+}\,\frac{3^2x^2}{7+n+\&c}$$

$$= 2\left(\frac{y}{m+1} - \frac{y^3}{m+3} + \frac{y^5}{m+5} - \&c\right)$$

$$\& \quad \frac{x}{2+n+}\,\frac{1.2\,x^2}{4+n+}\,\frac{2.3\,x^2}{6+n+}\,\frac{3.4\,x^2}{8+n+\&c}$$

$$= y - m\left(y+\frac{1}{y}\right)\left(\frac{y^2}{m+2} - \frac{y^4}{m+4} + \frac{y^6}{m+8} - \&c\right)$$

where $y = \dfrac{\sqrt{1+x^2}-1}{x}$ and $m = \dfrac{n}{\sqrt{1+x^2}}$.

$$2^p\left\{\frac{1}{n+p} - \frac{p}{\lfloor 1}\cdot\frac{1}{n+p+2} + \frac{p(p+1)}{\lfloor 2}\cdot\frac{1}{n+p+4} - \&c\right\}$$

$$= \frac{1}{n+}\,\frac{1.p}{n+}\,\frac{2(p+1)}{n+}\,\frac{3(p+2)}{n+\&c}$$

$$\frac{x}{p+n+}\,\frac{1.p\,x^2}{p+2+n+}\,\frac{2(p+1)x^2}{p+4+n+}\,\frac{3(p+2)x^2}{p+6+n+\&c}$$

$$= \left(1+\frac{1}{x^2}\right)^{\frac{p-1}{2}}(2y)^p\left\{\frac{1}{m+p} - \frac{p}{\lfloor 1}\cdot\frac{y^2}{m+p+2} + \frac{p(p+1)}{\lfloor 2}\cdot\frac{y^4}{m+p+4} - \&c\right\}$$

where $y = \dfrac{\sqrt{1+x^2}-1}{x}$ and $m = \dfrac{n}{\sqrt{1+x^2}}$.

$$\frac{\pi}{2}\int_0^\infty \frac{dx}{e^{x^n}+e^{-x^n}}$$

$$= \sqrt[n]{\frac{\pi}{2}}\;\Gamma\left(\frac{1}{n}\right)\cos\frac{\pi}{2n}\int_0^\infty \frac{x^{n-2}\,dx}{e^{x^n}+e^{-x^n}}$$

$$+\ \left|\frac{1}{n}-1\right.$$

$$\frac{x}{4n+2+}\;\frac{x^2}{4n+6+}\;\frac{x^2}{4n+10+}\;\&c$$

$$+\ \frac{2n}{x+}\;\frac{n-1}{1-}\;\frac{n+1}{x+}\;\frac{n-2}{1-}\;\frac{n+2}{x-}\;\&c$$

$= 1$ nearly.

$$\frac{1}{x+}\;\frac{a}{x+}\;\frac{a^3}{x+}\;\frac{a^5}{x+\&c} = 1-\frac{ax}{1+a+}\;\frac{a^3x}{1+a^2-}\;\frac{a^2x}{1+a^3+}\;\frac{a^6x}{1+a^4-}\;\frac{a^3x}{1+a^5+}\;\frac{a^9x}{1+a^6-}\;\&c$$

nearly.

$$\frac{1}{1-}\;\frac{a}{1+a-}\;\frac{a^3}{1+a^2-}\;\frac{a^5}{1+a^3-}\;\&c$$

$$= \frac{1-a^2}{1-a}\cdot\frac{1-a^5}{1-a^4}\cdot\frac{1-a^8}{1-a^7}\;\&c$$

$$\frac{1-a^3}{1-a}\cdot\frac{1-a^7}{1-a^5}\cdot\frac{1-a^{11}}{1-a^9}\;\&c = \frac{1}{1-}\;\frac{a}{1+a^2-}\;\frac{a^3}{1+a^4-}\;\frac{a^5}{1+a^6}$$

$$\frac{1}{1+}\;\frac{x}{1+}\;\frac{x^2+x}{1+}\;\frac{x^3}{1+}\;\frac{x^4+x^2}{1+}\;\frac{x^5}{1+}$$

$$= \frac{\psi(x^2)}{\psi(x)}.$$

$$\frac{f(-x,-x^7)}{f(-x^3,-x^5)} = \cfrac{1}{1+\cfrac{x+x^2}{1+\cfrac{x^4}{1+\cfrac{x^3+x^6}{1+\cfrac{x^8}{1+\&c}}}}}$$

$$m\left\{\frac{1}{2(m^2+n^2)}+\frac{1}{m^2+(1+n)^2}+\frac{1}{m^2+(2+n)^2}+\&c\right\}$$

$$+n\left\{\frac{1}{2(m^2+n^2)}+\frac{1}{n^2+(1+m)^2}+\frac{1}{n^2+(2+m)^2}+\&c\right\}-$$

$$=\frac{\pi}{2}+\frac{mn}{\pi(m^2+n^2)^2}+4mn\left\{\frac{1}{e^{2\pi}-1}\left(\frac{1}{m^2+\overline{1+n}^2}\cdot\frac{1}{m^2+\overline{1-n}^2}+\frac{1}{n^2+\overline{1+m}^2}\cdot\frac{1}{n^2+\overline{1-m}^2}\right)\right.$$

$$\left.+\frac{2}{e^{4\pi}-1}\left(\frac{1}{m^2+\overline{2+n}^2}\cdot\frac{1}{m^2+\overline{2-n}^2}+\frac{1}{n^2+\overline{2+m}^2}\cdot\frac{1}{n^2+\overline{2-m}^2}\right)+\&c\right\}$$

$$-2\pi.\frac{1-e^{2\pi m}\cos 2\pi n-e^{2\pi n}\cos 2\pi m+e^{2\pi(m+n)}\cos 2\pi(m-n)}{(e^{4\pi m}-2e^{2\pi m}\cos 2\pi n+1)(e^{4\pi n}-2e^{2\pi n}\cos 2\pi m+1)}$$

If $\phi(\alpha,\beta) = \alpha\left\{\frac{1}{2(\alpha^2+\beta^2)} + \frac{1}{\alpha^2+(1+\beta)^2} + \frac{1}{\alpha^2+(2+\beta)^2} + \&c\right\}$

$$- 4\alpha\beta\left\{\frac{1}{e^{2\pi}-1}\cdot\frac{1}{\alpha^2+\overline{1+\beta}^2}\cdot\frac{1}{\alpha^2+\overline{1-\beta}^2} + \frac{2}{e^{4\pi}-1}\cdot\frac{1}{\alpha^2+\overline{2+\beta}^2}\cdot\frac{1}{\alpha^2+\overline{2-\beta}^2}\right\}$$

$$+ \frac{\pi}{e^{4\pi\alpha} - 2e^{2\pi\alpha}\cos 2\pi\beta + 1}.$$

then $\phi(\alpha,\beta) + \phi(\beta,\alpha) = \frac{\pi}{2} + \frac{\alpha\beta}{\pi(\alpha^2+\beta^2)^2}$

$$+ \frac{\pi}{2}\cdot\frac{\cosh 2\pi(\alpha-\beta) - \cos 2\pi(\alpha-\beta)}{(\cosh 2\pi\alpha - \cos 2\pi\beta)(\cosh 2\pi\beta - \cos 2\pi\alpha)}$$

If $\phi(\alpha,\beta) = \alpha\left\{\frac{1}{\alpha^2+(1+\beta)^2} + \frac{1}{\alpha^2+(3+\beta)^2} + \frac{1}{\alpha^2+(5+\beta)^2} + \&c\right\}$

$$+ 4\alpha\beta\left\{\frac{1}{e^{\pi}+1}\cdot\frac{1}{\alpha^2+\overline{1+\beta}^2}\cdot\frac{1}{\alpha^2+\overline{1-\beta}^2} + \&c\right\}$$

$$+ \frac{\pi/2}{e^{2\pi\alpha} + 2e^{\pi\alpha}\cos\pi\beta + 1},$$ then

$$\phi(\alpha,\beta) + \phi(\beta,\alpha) = \frac{\pi}{4} + \frac{\pi}{4}\cdot\frac{\cosh\pi(\alpha-\beta) - \cos\pi(\alpha-\beta)}{(\cosh\pi\alpha + \cos\pi\beta)(\cosh\pi\beta + \cos\pi\alpha)}$$

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$$\frac{1}{4}+\frac{1}{1+(2n)^2}\cdot\frac{1}{e^{\pi m}+e^{-\pi m}}+\frac{1}{1+(4n)^2}\cdot\frac{1}{e^{2\pi m}+e^{-2\pi m}}$$

$$+m\left\{\frac{1}{n^2-m^2}\cdot\frac{1}{e^{\frac{\pi}{m}}-1}-\frac{1}{(3n)^2-m^2}\cdot\frac{1}{e^{\frac{3\pi}{m}}-1}+\&c\right\}$$

$$=\frac{\pi}{4n}\cdot\frac{\sec\frac{\pi m}{2n}}{e^{\frac{\pi}{n}}-1}+\frac{1}{2}\left(\frac{1}{n+m}-\frac{1}{3n+m}+\frac{1}{5n+m}-\&c\right)$$

$$\frac{x}{8\pi}+\frac{\sin x}{1(e^{2\pi}-1)}+\frac{\sin 4x}{2(e^{4\pi}-1)}+\frac{\sin 9x}{3(e^{6\pi}-1)}+$$

$$=\frac{1}{4}\left\{\frac{B_2}{1\lfloor 1}x-\frac{B_6}{3\lfloor 3}x^3+\frac{B_{10}}{5\lfloor 5}x^5-\&c\right\}$$

If $\alpha\beta=\frac{\pi^2}{2}$

$$\alpha^2\left\{\frac{\operatorname{sech}\frac{\pi}{2}}{\cosh\alpha+\cos\alpha}-\frac{3^3\operatorname{sech}\frac{3\pi}{2}}{\cosh 3\alpha+\cos 3\alpha}+\&c\right\}$$

$$=\beta^2\left\{\frac{\operatorname{sech}\frac{\pi}{2}}{\cosh\beta+\cos\beta}-\frac{3^3\operatorname{sech}\frac{3\pi}{2}}{\cosh 3\beta+\cos 3\beta}+\&c\right.$$

$$\int_0^\infty\frac{\sin 2nx}{x(\cosh\pi x+\cos\pi x)}$$

$$=\frac{\pi}{4}-2\left\{\frac{e^{-n}\cos n}{\cosh\frac{\pi}{2}}-\frac{e^{-3n}\cos 3n}{3\cosh\frac{3\pi}{2}}+\&c\right\}$$

If $\alpha\beta=\frac{\pi^2}{4}$, then

$$\frac{1}{\cosh\alpha+\cos\alpha}-\frac{1}{3(\cosh 3\alpha+\cos 3\alpha)}+\&c$$

$$+\frac{2\cos\beta\cosh\beta}{\cosh\frac{\pi}{2}(\cosh 2\beta+\cos 2\beta)}-\frac{2\cos 3\beta\cosh 3\beta}{3\cosh\frac{3\pi}{2}(\cosh 6\beta+\cos 6\beta)}$$

$$+\frac{2\cos 5\alpha\cosh 5\beta}{5\cosh\frac{5\pi}{2}(\cosh 10\beta+\cos 10\beta)}-\&c=\frac{\pi}{8}.$$ ~~180~~ 168

If $\alpha\beta=\frac{\pi^2}{2}$, then

$$\frac{\cos\alpha}{\cosh\alpha-\cos\alpha}-\frac{\cos 3\alpha}{3(\cosh 3\alpha-\cos 3\alpha)}+\&c$$

$$=\frac{\pi^3}{32\alpha^2}-\frac{\pi}{8}+\frac{\sin\beta\sinh\beta}{\cosh 2\beta+\cos 2\beta}\cdot\frac{\coth\pi}{1}$$

$$+\frac{\sin 2\beta\sinh 2\beta}{\cosh 4\beta+\cos 4\beta}\cdot\frac{\coth 2\pi}{2}+\&c$$

$$\frac{B_2}{1.2.2n}+\frac{B_4}{3.4.2^2n^3}-\frac{B_6}{5.6.2^3n^5}-\frac{B_8}{7.8.2^4n^7}+\&c$$

$$=~~n+\frac{\pi}{4}n+\log\lfloor n~~$$

$$=\log\frac{e^n\lfloor n}{n^n\sqrt{2\pi n}}+n\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)$$

$$-\frac{1}{2}\log\left[\sqrt{2}\left\{1+\left(\frac{n}{n+1}\right)^2\right\}\left\{1+\left(\frac{n}{n+2}\right)^2\right\}\&c\right].$$

$$\left\{1+\left(\frac{n}{1}\right)^3\right\}\left\{1+\left(\frac{n}{2}\right)^3\right\}\left\{1+\left(\frac{n}{3}\right)^3\right\}\&c$$

$$\times\left\{1+3.\left(\frac{n}{n+2}\right)^2\right\}\left\{1+3.\left(\frac{n}{n+4}\right)^2\right\}\left\{1+3.\left(\frac{n}{n+6}\right)^2\right\}\&c$$

$$=\frac{\lfloor\frac{n}{2}-1}{\lfloor\frac{n-1}{2}}\cdot\frac{\cosh\pi n\sqrt{3}-\cos\pi n}{2^{n+2}\cdot\pi n\sqrt{\pi}}.$$

~~$\frac{1}{2}\log(2\pi x) + \frac{1}{3}\log\left(1+\frac{x^3}{1^3}\right)\left(1+\frac{x^3}{2^3}\right)\left(1+\frac{x^3}{3^3}\right)$ &c~~

~~$= \frac{2\pi x}{3\sqrt{3}} + \frac{B_4}{3.4x^3} - \frac{B_{10}}{9.10x^9} + \frac{B_{16}}{15.16.x^{15}}$ &c~~

$\frac{1}{2}\log(2\pi n) + \frac{1}{3}\log\left(1+\frac{n^3}{1^3}\right)\left(1+\frac{n^3}{2^3}\right)\left(1+\frac{n^3}{3^3}\right)$ &c

$= \frac{1}{3}\log\left(e^{\pi n\sqrt{3}} - 2\cos\pi n + e^{-\pi n\sqrt{3}}\right)$

$- \frac{\pi n}{3\sqrt{3}} + \frac{B_4}{3.4n^3} - \frac{B_{10}}{9.10n^9} + $ &c

$\left\{1+\left(\frac{m+n}{1+m}\right)^3\right\}\left\{1+\left(\frac{m+n}{2+m}\right)^3\right\}\left\{1+\left(\frac{m+n}{3+m}\right)^3\right\}$ &c

$\times\left\{1+\left(\frac{m+n}{1+n}\right)^3\right\}\left\{1+\left(\frac{m+n}{2+n}\right)^3\right\}\left\{1+\left(\frac{m+n}{3+n}\right)^3\right\}$ &c

$= \frac{(\underline{|m})^3(\underline{|n})^3}{\underline{|2m+n}\ \underline{|2n+m}} \cdot \frac{\cosh\pi(m+n)\sqrt{3} - \cos\pi(m-n)}{2\pi^2(m^2+mn+n^2)}$

$\int_0^\infty \log(1+x^2)\cos nx\,dx = -\frac{\pi}{n}e^{-n}$

$\log\left(1+\frac{x^2}{1^2}\right) - 3\log\left(1+\frac{x^2}{3^2}\right) + 5\log\left(1+\frac{x^2}{5^2}\right) -$ &c

$= \frac{4}{\pi}\cdot\left\{\frac{1-e^{-\frac{\pi x}{2}}}{1^2} - \frac{1-e^{-\frac{3\pi x}{2}}}{3^2} + \frac{1-e^{-\frac{5\pi x}{2}}}{5^2} - \text{\&c}\right\}$

$- 2x\tan^{-1}e^{-\frac{\pi x}{2}}.$

$$\log\left(1-\frac{x^2}{1^2}\right)-3\log\left(1-\frac{x^2}{3^2}\right)+5\log\left(1-\frac{x^2}{5^2}\right)-\&c$$

$$=\frac{4}{\pi}\left\{\frac{1-\cos\frac{\pi x}{2}}{1^2}-\frac{1-\cos\frac{3\pi x}{2}}{3^2}+\&c\right\}$$

$$+ \log \frac{\pi-\pi x}{4}$$

$$=\frac{4}{\pi}\left\{\frac{1-\tan\left(\frac{\pi}{4}-\frac{\pi x}{4}\right)}{1^2}-\frac{1-\tan^3\left(\frac{\pi}{4}-\frac{\pi x}{4}\right)}{3^2}+\&c\right\}$$

$$+\log\tan\frac{\pi-\pi x}{4}.$$

If $\frac{\pi\alpha}{2}=\log\tan\left(\frac{\pi}{4}+\frac{\pi\beta}{4}\right)$

then $\log\left(1+\frac{\alpha^2}{1^2}\right)-3\log\left(1+\frac{\alpha^2}{3^2}\right)+5\log\left(1+\frac{\alpha^2}{5^2}\right)-\&c$

$$=\frac{\pi\alpha\beta}{2}+\log\left(1-\frac{\beta^2}{1^2}\right)-3\log\left(1-\frac{\beta^2}{3^2}\right)+5\log\left(1-\frac{\beta^2}{5^2}\right)-\&c$$

$$\int_0^\infty\frac{\sinh ax}{\sinh\pi x}\cdot\frac{dx}{n^2+x^2}=\int_0^1\frac{x^n}{n}\cdot\frac{\sin a\,dx}{1+2x\cos a+x^2}$$

$$\int_0^\infty\frac{\sin ax}{\sinh\frac{\pi x}{2}}\cdot\frac{dx}{1+x^2}=\frac{\pi}{2}\sinh a$$

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$$\frac{\phi^2(e^{-\pi\sqrt{\frac{n}{3}}})}{\phi^2(e^{-\pi\sqrt{3n}})} = \sqrt{3}\, a_n$$

$a_1 = 1;\ a_5 = \frac{\sqrt{5}-1}{2};\ a_7 = \frac{\sqrt{3}+1}{\sqrt{2}}\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{\frac{3}{2}}.$

$$\frac{\phi^4(e^{-\pi\sqrt{\frac{n}{3}}})}{\phi^4(e^{-\pi\sqrt{3n}})} = 3a_n \cdot \psi^2(e^{-5\pi}$$

$a_1 = 1;\ a_5 = \left(\frac{\sqrt{5}-1}{2}\right)^2;\ a_7 = (2+\sqrt{3})\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3;$

$a_{11} = (10-3\sqrt{11})(2\sqrt{3}+\sqrt{11});\ a_{13} =$

$$n\, e^{-\frac{\pi}{4}(n-1)\sqrt{\frac{m}{n}}}\ \frac{\psi^2(e^{-\pi\sqrt{mn}})\,\phi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(e^{-\pi\sqrt{\frac{m}{n}}})\,\phi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}$$

$= a_{m,n}.\ a_{3,5} = \frac{3-\sqrt{5}}{2};\ a_{3,7} = 2-\sqrt{3};\ a_{3,11} = 2\sqrt{3}-\sqrt{11},$

$a_{3,13} = \left(\sqrt{\frac{5+\sqrt{13}}{8}} - \sqrt{\frac{\sqrt{13}-3}{8}}\right)^8;\ a_{3,19} = 2\sqrt{19} - 5\sqrt{3}.$

$a_{3,23} = \left(\sqrt{\frac{7+4\sqrt{3}}{2}} - \sqrt{\frac{5+4\sqrt{3}}{2}}\right)^2;\ a_{3,31} = (2-\sqrt{3})^3.$

$a_{3,59} = 102\sqrt{3} - 23\sqrt{59};\ a_{3,71} = \left(\sqrt{\frac{175+100\sqrt{3}}{2}} - \sqrt{\frac{173+100\sqrt{3}}{2}}\right)^2$

$a_{5,9} = (2-\sqrt{3})^2;\ a_{5,11} = \left(\sqrt{\frac{7+\sqrt{5}}{8}} - \sqrt{\frac{\sqrt{5}-1}{8}}\right)^8;$

$a_{5,13} = \left(\sqrt{\frac{9+\sqrt{65}}{2}} - \sqrt{\frac{7+\sqrt{65}}{2}}\right)^2;\ a_{5,17} = (\sqrt{17}-4)^2;$ &

$a_{5,29} = \left(\sqrt{49+4\sqrt{145}} - \sqrt{48+4\sqrt{145}}\right)^2;$

$a_{7,9} = \left(\sqrt{\frac{5+\sqrt{21}}{8}} - \sqrt{\frac{\sqrt{21}-3}{8}}\right)^8$

$a_{3,3} = \frac{1}{\sqrt{3}}$; $a_{3,9} = \frac{1}{(\sqrt[3]{2}+1)^2}$; $a_{3,15} = \frac{2-\sqrt{3}}{3}$ ~~18~~ 170

~~$a_{3,}$~~

$$\frac{\phi^2(e^{-\pi})}{\phi^2(e^{-n\pi})} = n\, b_n \sqrt[8]{4\beta(1-\beta)}$$

$b_1 = 1$; $b_3 = \frac{1}{\sqrt[4]{3}}$; $b_5 = 1$; $b_7 = \frac{\sqrt{\frac{3+\sqrt{7}}{\sqrt{2}}}}{\sqrt[8]{7}}$.

~~$b_{17} =$~~

$$\frac{1^2 x}{1+x} - \frac{2^2 x^3}{1+x^2} + \frac{3^2 x^6}{1+x^3} - \frac{4^2 x^{10}}{1+x^4} + \&c$$

$$= \phi^2(-x)\left\{x \cdot \frac{1+x}{(1-x)^2} + x^6 \cdot \frac{1+x^3}{(1-x^3)^2} + x^{10}\frac{1+x^5}{(1-x^5)^2} + \&c\right\}$$

$$\frac{1+x}{1-x} - 3^2 x^2 \cdot \frac{1+x^3}{1-x^3} + 5^2 x^6 \cdot \frac{1+x^5}{1-x^5} - \&c$$

$$= \psi^2(x)\left\{1 - \frac{8x^2}{(1+x)^2} + \frac{8x^6}{(1+x^2)^2} - \frac{8x^{12}}{(1+x^3)^2} + \&c\right\}$$

$$x\psi(x)\psi(x^4) = \frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^6}{1-x^5} - \frac{x^{10}}{1-x^7} + \&c$$

$$\frac{x^n}{\lfloor n}\left\{1 + \frac{x^2}{\lfloor 1} \cdot \frac{1}{1+n} + \frac{x^4}{\lfloor 2} \cdot \frac{1}{(1+n)(2+n)} + \&c\right\}$$

$$- \frac{x^{-n}}{\lfloor -n}\left\{1 + \frac{x^2}{\lfloor 1} \cdot \frac{1}{1-n} + \frac{x^4}{\lfloor 2} \cdot \frac{1}{(1-n)(2-n)} + \&c\right\}$$

If n is a positive integer

$$\frac{1^{4n}}{(e^{\pi}-e^{-\pi})^2}+\frac{2^{4n}}{(e^{2\pi}-e^{-2\pi})^2}+\&c=\frac{n}{\pi}\left(\frac{B_{4n}}{8n}+\frac{1^{4n-1}}{e^{2\pi}-1}+\frac{2^{4n-1}}{e^{4\pi}-1}+\right)$$

$$\frac{1}{2x^2}+\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\&c$$

$$=\frac{1}{2\pi x^3}+\frac{\pi}{3x}-\frac{\pi^2}{\sin^2\pi x\,(e^{2\pi x}-1)}$$

$$+4x\left\{\frac{1}{e^{2\pi}-1}\cdot\frac{1}{(1^2-x^2)^2}+\frac{2}{e^{4\pi}-1}\cdot\frac{1}{(2^2-x^2)^2}+\&c\right\}$$

$$+8\pi x^3\left\{\frac{1}{(e^{\pi}-e^{-\pi})^2}\cdot\frac{1}{1^4-x^4}+\frac{1}{(e^{2\pi}-e^{-2\pi})^2}\cdot\frac{1}{2^4-x^4}+\&c\right\}$$

If $\theta\frac{u}{\sqrt{2}}=v+\frac{1}{2}\cdot\frac{v^5}{5}+\frac{1.3}{2.4}\cdot\frac{v^9}{9}+\&c$, then

$$\frac{u^2}{2\theta^2}=\frac{1}{\sin^2\theta}-\frac{1}{\pi}-8\left(\frac{\cos 2\theta}{e^{2\pi}-1}+\frac{2\cos 4\theta}{e^{4\pi}-1}+\&c\right).$$

$$\cot\theta+\frac{\theta}{\pi}+4\left(\frac{\sin 2\theta}{e^{2\pi}-1}+\frac{\sin 4\theta}{e^{4\pi}-1}+\&c\right)$$

$$=\frac{u}{\sqrt{2}}\left\{\frac{1}{v}-\frac{1}{2}\cdot\frac{v^3}{3}-\frac{1.3}{2.4}\cdot\frac{v^7}{7}-\&c\right\}$$

$$\log\sin\theta+\frac{\theta^2}{2\pi}-2\left\{\frac{\cos 2\theta}{1(e^{2\pi}-1)}+\frac{\cos 4\theta}{2(e^{4\pi}-1)}+\&c\right\}$$

$$=\log\frac{v\sqrt{2}}{u}+\frac{1}{3}\cdot\frac{v^4}{4}+\frac{1.5}{3.7}\cdot\frac{v^8}{8}+\frac{1.5.9}{3.7.11}\cdot\frac{v^{12}}{12}+\&c$$

$$\frac{1}{2}\tan^{-1}v=\frac{\sin\theta}{\cosh\frac{\pi}{2}}+\frac{\sin 3\theta}{3\cosh\frac{3\pi}{2}}+\frac{\sin 5\theta}{5\cosh\frac{5\pi}{2}}+\&c$$

$$\frac{1}{4}\cos^{-1}v^2=\frac{\cos\theta}{\cosh\frac{\pi}{2}}-\frac{\cos 3\theta}{3\cosh\frac{3\pi}{2}}+\frac{\cos 5\theta}{5\cosh\frac{5\pi}{2}}-\&c$$

$$\frac{\pi\theta}{8}-\frac{\sin\theta}{1^2\cosh\frac{\pi}{2}}+\frac{\sin 3\theta}{3^2\cosh\frac{3\pi}{2}}-\&c$$

$$=\frac{\sqrt{2}}{4u}\left\{\frac{v^3}{3}+\frac{2}{3}\cdot\frac{v^7}{7}+\frac{2.4}{3.5}\cdot\frac{v^{11}}{11}+\&c\right\}.$$

If $\theta \frac{\mu}{2} = v - \frac{1}{2}\cdot\frac{v^5}{5} + \frac{1.3}{2.4}\cdot\frac{v^9}{9} - \&c$ 171 183

$$2\tan^{-1} v = \theta + \frac{\sin 2\theta}{\cosh \pi} + \frac{\sin 4\theta}{2\cosh 2\pi} + \frac{\sin 6\theta}{3\cosh 3\pi} + \&c$$

$$\frac{\pi}{8} - \frac{1}{2}\tan^{-1} v^2 = \frac{\cos\theta}{\cosh\frac{\pi}{2}} - \frac{\cos 3\theta}{3\cosh\frac{3\pi}{2}} + \frac{\cos 5\theta}{5\cosh\frac{5\pi}{2}} - \&c$$

$$\log\frac{1+v}{1-v} = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + 4\left\{\frac{\sin\theta}{e^{\pi}-1} - \frac{\sin 3\theta}{3(e^{3\pi}-1)} + \&c\right.$$

$$\frac{1}{2\pi x^4} + \coth\pi\left(\frac{1}{1+x^2+x^4} + \frac{1}{1-x^2+x^4}\right)$$

$$+ 2\coth 2\pi\left(\frac{1}{16+4x^2+x^4} + \frac{1}{16-4x^2+x^4}\right)$$

$$+ \quad \&c \qquad \&c$$

$$= \frac{\pi}{x^2\sqrt{3}}\cdot\frac{\sinh\pi x\sqrt{3}\,\sinh\pi x + \sin\pi x\sqrt{3}\,\sin\pi x}{(\cosh\pi x\sqrt{3} - \cos\pi x)(\cosh\pi x - \cos\pi x\sqrt{3})}$$

$$\Sigma\frac{1}{x} - \Sigma\frac{1}{x/3} + \frac{2}{x} - \log 3$$

$$= \frac{2}{3}\cdot\cfrac{1}{x^2 + \cfrac{2^3-2}{6 + \cfrac{4^3-4}{3x^2 + \cfrac{5^3-5}{6 + \cfrac{7^3-7}{5x^2 + \&c}}}}}$$

$$\frac{1^{6n}}{\left(e^{\frac{\pi\sqrt{3}}{2}} + e^{-\frac{\pi\sqrt{3}}{2}}\right)^2} - \frac{2^{6n}}{\left(e^{\pi\sqrt{3}} - e^{-\pi\sqrt{3}}\right)^2} + \&c$$

$$+ \frac{n\sqrt{3}}{\pi}\left\{\frac{B_{6n}}{12n}\cos 3\pi n - \left(\frac{1^{6n-1}}{e^{\pi\sqrt{3}}+1} - \&c\right)\right\} = 0$$

n being a positive integer

$$2^2\left\{1+240\left(\frac{1^3}{e^{2\pi\sqrt{2}}-1}+\frac{2^3}{e^{4\pi\sqrt{2}}-1}+\&c\right)\right\}-\left\{1+240\left(\frac{1^3}{e^{\pi\sqrt{2}}-1}+\frac{2^3}{e^{\pi 2\sqrt{2}}-1}\right)\right\}=0$$

$$2^3\left\{1-504\left(\frac{1^5}{e^{2\pi\sqrt{2}}-1}+\&c\right)\right\}+\left\{1-504\left(\frac{1^5}{e^{\pi\sqrt{2}}-1}+\&c\right)\right\}=0$$

$$2^4\left\{1+480\left(\frac{1^7}{e^{2\pi\sqrt{2}}-1}+\&c\right)\right\}-\left\{1+480\left(\frac{1^7}{e^{\pi\sqrt{2}}-1}+\&c\right)\right\}=0$$

If $S_n = \frac{B_n}{2n} + \frac{1^{n-1}}{e^{2\pi}-1} + \frac{2^{n-1}}{e^{4\pi}-1} + \&c$ where n is a multiple of 4,

then
$$\frac{(n+3)(n-4)}{24} S_{n+2} = \frac{n(n-1)(n-2)(n-3)}{\lfloor 4} S_4 S_{n-2} + \frac{n(n-1) \cdots (n-7)}{\lfloor 8} S_8 S_{n-6} + \&c$$

If $\frac{2}{3}\theta u = v + \frac{1}{2}\cdot\frac{v^7}{7} + \frac{1.3}{2.4}\cdot\frac{v^{13}}{13} + \&c$

then
$$\frac{4}{9}\frac{u^2}{v^2} = \frac{1}{\sin^2\theta} - \frac{2}{\pi\sqrt{3}} + 8\left(\frac{\cos 2\theta}{e^{\pi\sqrt{3}}+1} - \frac{2\cos 4\theta}{e^{2\pi\sqrt{3}}-1} + \&c\right)$$

where $u = \frac{\sqrt{\pi}}{\lfloor -\frac{1}{6}\ \lfloor -\frac{1}{3}}$.

$$\frac{1}{2}\log\left[\left\{1+\left(\frac{x}{n+1}\right)^2\right\}\left\{1+\left(\frac{x}{n+2}\right)^2\right\}\left\{1+\left(\frac{x}{n+3}\right)^2\right\}\&c\right]$$
$$= \log \lfloor n + n + x\tan^{-1}\frac{x}{n} - \frac{n}{2}\log(n^2+x^2) - \frac{1}{2}\log\left(2\pi\sqrt{n^2+x^2}\right) - \int_0^\infty \frac{\tan^{-1}\frac{2nz}{n^2+x^2-z^2}}{e^{2\pi z}-1}\,dz.$$

$$\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots\cdots+\sqrt{x}$$
$$=C+\frac{2}{3}x\sqrt{x}+\frac{1}{2}\sqrt{x}+\frac{1}{6}\left\{\frac{1}{(\sqrt{x}+\sqrt{x+1})^3}+\frac{1}{(\sqrt{x+1}+\sqrt{x+2})^3}+\&c\right\}$$

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$$1\sqrt{1}+2\sqrt{2}+3\sqrt{3}+\cdots\cdots+x\sqrt{x}$$
$$=C+\frac{2}{5}x^2\sqrt{x}+\frac{x}{2}\sqrt{x}+\frac{\sqrt{x}}{8}+\frac{1}{40}\left\{\frac{1}{(\sqrt{x}+\sqrt{x+1})^5}+\frac{1}{(\sqrt{x+1}+\sqrt{x+2})^5}+\&c\right\}$$

$$(1^2\sqrt{1}+2^2\sqrt{2}+3^2\sqrt{3}\quad\cdots\quad+x^2\sqrt{x})+\frac{1}{16}(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{x})$$
$$=C+\frac{2}{7}x^2\sqrt{x}+\frac{x^2}{2}\sqrt{x}+\frac{x\sqrt{x}}{4}+\frac{\sqrt{x}}{32}+\frac{1}{224}\left\{\frac{1}{(\sqrt{x}+\sqrt{x+1})^7}+\&c\right\}$$

~~$1^3\sqrt{1}+2^3\sqrt{2}\cdots+x^2\sqrt{x}+$~~

$$C+\frac{\pi}{3}\log x+\frac{1}{2x}-\frac{1}{4\pi x^2}+2\left(\frac{1}{e^{2\pi}-1}\cdot\frac{1}{1^2-x^2}+\frac{2}{e^{4\pi}-1}\cdot\frac{8}{2^2-x^2}+\&c\right)$$
$$+\frac{\pi\cot\pi x}{e^{2\pi x}-1}+\frac{2\pi\log(2\sin\pi x)}{(e^{\pi x}-e^{-\pi x})^2}-2\pi\left\{\frac{\log(1^4-x^4)}{(e^{\pi}-e^{-\pi})^2}+\frac{\log(2^4-x^4)}{(e^{2\pi}-e^{-2\pi})^2}+\&c\right.$$
$$-2\pi\sum_{n=1}^{n=\infty}e^{-2\pi nx}\left[n^2\left\{\frac{\sin 2\pi x}{1^2+n^2}+\frac{\sin 4\pi x}{2^2+n^2}+\&c\right\}\right.$$
$$\left.-n^3\left\{\frac{\cos 2\pi x}{1(1^2+n^2)}+\frac{\cos 4\pi x}{2(2^2+n^2)}+\&c\right\}\right]$$
$$=1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{x}.$$

$y\sqrt[4]{\frac{\sqrt{13}-3}{2}}$ where $\sqrt{5}\left(y^3+y^2\frac{\sqrt{13}-1}{2}+y\cdot\frac{\sqrt{13}+1}{2}-1\right)$
$$\pm\left\{y^3+y^2\left(\frac{\sqrt{13}+1}{2}\right)^2+y\cdot\left(\frac{\sqrt{13}-1}{2}\right)^2+1\right\}=0$$

$$\sqrt{765}\qquad(\sqrt{5}-2)^8\left(\frac{\sqrt{85}-9}{2}\right)^6(4-\sqrt{15})^6(16-\sqrt{255})^2$$
$$\times\left(\sqrt{\frac{22+3\sqrt{51}}{4}}-\sqrt{\frac{18+3\sqrt{51}}{4}}\right)^{12}\left(\sqrt{\frac{10+\sqrt{51}}{4}}-\sqrt{\frac{6+\sqrt{51}}{4}}\right)^{12}$$

If $\phi = \frac{x}{1+}\frac{x^5}{1+}\frac{x^{10}}{1+}$ &c. & $f = \frac{x^{\frac{1}{5}}}{1+}\frac{x}{1+}\frac{x^2}{1+}\frac{x^3}{1+}$ &c.

then $f^5 = \phi \cdot \frac{1-2\phi+4\phi^2-3\phi^3+\phi^4}{1+3\phi+4\phi^2+2\phi^3+\phi^4}$

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$$\sqrt{505}\ (\sqrt{5}-2)^{14}(\sqrt{101}-10)^{6}\left(\frac{5\sqrt{5}+\sqrt{101}}{4}\pm\sqrt{\frac{105+5\sqrt{505}}{8}}\right)$$

$\sqrt{38}$ $$\frac{g^3+g\sqrt{2}}{1+g^2\sqrt{2}}=\sqrt{1+\sqrt{2}}$$

$\sqrt{26}$ $$\frac{g^3+g\cdot\frac{\sqrt{13}+1}{2}}{1+g^2\cdot\frac{\sqrt{13}-1}{2}}=\sqrt{\frac{3+\sqrt{13}}{2}}$$

$\sqrt{50}$ $$\frac{g^3-g^2}{1+g}=\frac{\sqrt{5}+1}{2}$$

$$\frac{a}{x+}\frac{a^4}{x+}\frac{a^8}{x+}\frac{a^{12}}{x+}\text{\&c} = 1-\frac{ax}{1+}\frac{a^2}{1-}\frac{a^3x}{1+}\frac{a^4}{1-}\text{\&c}$$ nearly.

$(\frac{1}{64})$. $\sqrt{23}$. $1-g^3 = g^2$
$\sqrt{31}$ $1-g^3 = g$

(64.) $\sqrt{11}$ $\frac{1}{2}-g^3 = g-g^2$
$\sqrt{19}$ $\frac{1}{2}-g^3 = g^2$
$\sqrt{27}$ $\frac{1}{2}-g^3 = g^2\sqrt[3]{3}$.
$\sqrt{43}$ $\frac{1}{2}-g^3 = g$ $\quad\sqrt{67}$ $\frac{1}{2}-g^3 = g^2+g$

$\sqrt{3}$ $\quad 2-\sqrt{3}$

$\sqrt{5}$ $\quad \left(\sqrt{\frac{3+\sqrt{5}}{4}} - \sqrt{\frac{\sqrt{5}-1}{4}}\right)^4$

$\sqrt{7}$ $\quad 8-3\sqrt{7}$

$\sqrt{9}$ $\quad \left(\sqrt{\frac{3+\sqrt{3}}{4}} - \sqrt{\frac{\sqrt{3}-1}{4}}\right)^8$

$\sqrt{13}$ $\quad \left(\sqrt{\frac{7+\sqrt{13}}{4}} - \sqrt{\frac{3+\sqrt{13}}{4}}\right)^4$

$\sqrt{15}$ $\quad (2-\sqrt{3})^2(4-\sqrt{15})$

$\sqrt{17}$ $\quad \left(\sqrt{\frac{3+\sqrt{4+\sqrt{17}}}{4}} - \sqrt{\frac{\sqrt{4+\sqrt{17}}-1}{4}}\right)^8$

$\sqrt{21}$

$\sqrt{25}$ $\quad \left(\sqrt{\frac{5+\sqrt{5}}{4}} - \sqrt{\frac{\sqrt{5}+1}{4}}\right)^8$

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$\sqrt{7}$ ——— $8 - 3\sqrt{7}$

$\sqrt{15}$ ——— $(2-\sqrt{3})^2 (4-\sqrt{15})$

$\sqrt{39}$ ———

$\sqrt{55}$ ——— $(10-3\sqrt{11})(3\sqrt{5}-2\sqrt{11})\left(\sqrt{\frac{5+\sqrt{5}}{2}} - \sqrt{\frac{1+\sqrt{5}}{2}}\right)$ 4